Supplementary Material of
Smooth Representation Clustering

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In this document, we prove Proposition 2 and Proposition 3 in detail.

To prove Proposition 2, we first provide two lemmas:

Lemma S.1 [1]: *Given a subspace S spanned by a set of orthogonal basis $[\mathbf{u}_1, \ldots, \mathbf{u}_r]$ ($\mathbf{u}_i \in \mathbb{R}^{n\times 1}$) and its orthogonal complement $S_\perp$, for any matrix $M \in \mathbb{R}^{n\times k}, \forall k$, there exist a unique pair $M_1 \in S$ and $M_2 \in S_\perp$ such that*

$$M = M_1 + M_2. \tag{1}$$

Lemma S.2: *Let A and B be matrices of the same size. If $AB^T = 0$ and $A^T B = 0$, then $\|A + B\|_* = \|A\|_* + \|B\|_*$.*

Proof: Note the singular value decompositions (SVDs) of A and B as:

$$A = U_A \Sigma_A V_A^T,\ B = U_B \Sigma_B V_B^T, \tag{2}$$

where U_A and U_B are left-invertible; and V_A and V_B are right-invertible. From the condition $AB^T = 0$, we get $V_A^T V_B = 0$. Similarly, $A^T B = 0$ implies $U_A^T U_B = 0$. Hence,

$$A + B = \begin{bmatrix} U_A & U_B \end{bmatrix} \begin{bmatrix} \Sigma_A & \\ & \Sigma_B \end{bmatrix} \begin{bmatrix} V_A & V_B \end{bmatrix}^T \tag{3}$$

is a valid SVD of $A + B$. It is easy to check that $\|A + B\|_* = \|A\|_* + \|B\|_*$. □

Proposition 2: *The LRR problem (4) [2] has a unique optimal solution.*

$$\min_Z\ f(Z) = \alpha\|X - XZ\|_F^2 + \|Z\|_*. \tag{4}$$

Proof: Note the SVD of X as $X = U\Sigma V^T$ with $U \in \mathbb{R}^{d\times r}$, $\Sigma = \text{diag}(\mathbf{s})$ ($\mathbf{s}_i > 0, \forall 1 \leq i \leq r$) and $V \in \mathbb{R}^{n\times r}$. Note S as the subspace spanned by columns of V, and $S_\perp$ as the orthogonal complement of S.

Suppose Z^* is an optimal solution of problem (4). According to Lemma S.1, there exist a unique pair $Z_1^* \in S$ and $Z_2^* \in S_\perp$ that $Z^* = Z_1^* + Z_2^*$. Next we prove that Z_2^* must equal 0.

Suppose $Z_2^* \neq 0$. We have $\|Z_2^*\|_* > 0$. The condition $\mathbf{Z}_2^* \in S_\perp$ implies $XZ_2^* = U\Sigma V^T Z_2^* = 0$. Then

$$\begin{aligned} f(Z^*) &= \alpha||X - XZ^*||_F^2 + ||Z||_* \\ &= \alpha||X - X(Z_1^* + Z_2^*)||_F^2 + ||Z_1^* + Z_2^*||_* \\ &= \alpha||X - XZ_1^*||_F^2 + ||Z_1^*||_* + ||Z_2^*||_* \\ &> f(Z_1^*) \end{aligned} \tag{5}$$

Equation (5) indicates Z_1^* is a better solution of problem (4) than Z^*, which is a contradiction. Hence $Z_2^* = 0$ is proved. As a result, we have $Z^* = Z_1^*$.

The condition $Z_1^* \in S$ indicates that there exists a unique matrix $W \in \mathbb{R}^{r\times n}$ that

$$Z_1^* = VW. \tag{6}$$

Substituting equation (6) into problem (4), we get a new optimization about W as

$$\min_W \; g(W) = \alpha\|X - XVW\|_F^2 + \|VW\|_* = \alpha\|X - U\Sigma W\|_F^2 + \|W\|_*. \tag{7}$$

It is easy to verify that the Hessian matrix of the first term

$$H_1 = I \otimes \Sigma U^T U \Sigma = I \otimes \Sigma^2 \succ 0, \tag{8}$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix, and $\otimes$ is the Kronecker product operator. According to equation (8), problem (7) is strictly convex and it has a unique solution W^*. This implies that the solution of problem (4), Z^*, is also unique, and $Z^* = VW^*$. □

Next we prove Proposition 3. Recall the optimization problem for self-representation based methods as (9).

$$\begin{aligned} \min_Z \quad & f(Z) = \alpha\|X - A(X)Z\|_l + \Omega(X, Z), \\ s.t. \quad & Z \in \mathcal{C}, \end{aligned} \tag{9}$$

Proposition 3: *Problems (9) with the following $\Omega(Z)$ and $\mathcal{C}$ have grouping effect:*

(1) $\Omega(Z) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}|Z_{ij}|^p\right)^q, p > 1, q \geq 0, \mathcal{C} = \emptyset.$

(2) $\Omega(Z) = tr((ZHZ^T)^p), H \succ 0, p \geq 1/2, \mathcal{C} = \emptyset.$

(3) $\Omega(Z) = tr((Z^T HZ)^p), H \succ 0, p \geq 1/2, \mathcal{C} = \emptyset.$

Proof: (1) It is easy to verify that EGE conditions (1) and (3) are satisfied.

Noting that the regularity term $\Omega(Z) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}|Z_{ij}|^p\right)^q = \sum_{j=1}^{n}\|Z_j\|_p^{pq}$, where $\|Z_j\|_p = \left(\sum_{i=1}^{n}|Z_{ij}|^p\right)^{1/p}$ is the ℓ_p vector-norm, we have $\Omega(Z)$ is strictly convex w.r.t Z. As a result, problem (9) has a unique solution. According to Proposition 1 in the paper, the grouping effect of this solution is also guaranteed.

(2) Regarding H defined by $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$ with $H(XP) = P^T H(X)P$, we can verify that $\Omega(Z) = \mathrm{tr}((ZHZ^T)^p)$ satisfy EGE conditions (1) and (3).

When $p > 1/2$, $\Omega(Z) = \mathrm{tr}((ZHZ^T)^p)$ is strictly convex w.r.t Z, and thus problem (9) has a unique solution. In the following, we will prove that when $p = 1/2$, problem (9) also has a unique solution

Since $H \succ 0$, we can find an invertible matrix $L \in \mathbb{R}^{n\times n}$ such that $H = LL^T$. Substituting $Z = YL^{-1}$ into $f(Z)$, we have

$$f(Z) = h(Y) = \alpha\|X - XYL^{-1}\|_F^2 + \mathrm{tr}((YY^T)^{1/2}). \tag{10}$$

Noting that $\|Y\|_* = \mathrm{tr}((YY^T)^{1/2})$, similar as the proof of Proposition 2, we conclude that $Y^* = VW \in S$ and thus an optimization problem w.r.t W is obtained as

$$\min_W \; g(W) = \alpha\|X - XVWL^{-1}\|_F^2 + \|VW\|_* = \alpha\|X - U\Sigma WL^{-1}\|_F^2 + \|W\|_*. \tag{11}$$

The Hessian matrix of the first term of $g(W)$ is

$$H_1 = (LL^T)^{-T} \otimes \Sigma^2 = H^{-T} \otimes \Sigma^2. \tag{12}$$

Since $H \succ 0$ and $\Sigma^2 \succ 0$, we get $H_1 \succ 0$, which indicates the uniqueness of the solution of problem (11). Hence, Problem (9) with $\Omega(Z) = \mathrm{tr}((ZHZ^T)^{1/2}), H \succ 0, \mathcal{C} = \emptyset$ also has a unique solution.

According to Proposition 1, the grouping effect is proved.

(3) When $p > 1/2$, the uniqueness and grouping effect of the solution can be easily proved.

In the following, we prove the proposition with $p = 1/2$. There exists a decomposition $H = LL^T, L \in \mathbb{R}^{n\times n}$. Substituting $Z = L^{-T}Y$ into $f(Z)$, we get

$$f(Z) = h(Y) = \alpha\|X - XL^{-T}Y\|_F^2 + \|Y\|_*. \tag{13}$$

Note $U\Sigma V^T$ as the SVD of XL^{-T} and S as the subspace spanned by the columns of XL^{-T}. Similarly as the proof of Proposition 2, we have $Y^* \in S$ and it is unique, which also implies the uniqueness of Z^*. As a result, Z^* has grouping effect. □

References

[1] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 2007. 1

[2] G. Liu, Z. Lin, and Y. Yu. Robust subspace segmentation by low-rank representation. In *ICML*, pages 663–670, 2010. 1