

# Supplemental Materials for Adaptive Partial Differential Equation Learning for Visual Saliency Detection

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## 1. General Statements

Before proving theoretical results, we first state our PDE system and its discretization as follows.

**LESD:**

$$\begin{aligned} \operatorname{div}(\mathbf{K}_{\mathbf{p}} \nabla f(\mathbf{p})) + \lambda(f(\mathbf{p}) - g(\mathbf{p})) = 0, \\ \text{s.t. } f(\mathbf{g}) = 0, f(\mathbf{p}) = s_{\mathbf{p}}, \mathbf{p} \in \mathcal{S}, \end{aligned} \quad (1)$$

where  $\lambda \geq 0$ .

**Discretization:**

$$f(\mathbf{p}) = \frac{1}{d_{\mathbf{p}} + \lambda} \left( \sum_{\mathbf{q} \in \mathcal{N}(\mathbf{p})} \mathbf{K}_{\mathbf{p}}(\mathbf{q}) f(\mathbf{q}) + \lambda g(\mathbf{p}) \right), \quad (2)$$

where  $d_{\mathbf{p}} = \sum_{\mathbf{q} \in \mathcal{N}(\mathbf{p})} \mathbf{K}_{\mathbf{p}}(\mathbf{q})$ ,  $\mathbf{K}_{\mathbf{p}}(\mathbf{q}) \geq 0$  and  $g(\mathbf{p}) \geq 0$ .

Then the main theoretical results (Theorem 1 in Section 2 and Corollary 2 in Section 3) in our manuscript are listed.

**Theorem 1** *Let  $f(\mathbf{p}; \mathcal{S})$  be the visual attention score for image element  $\mathbf{p}$  and the sources  $\{s_{\mathbf{p}} \geq 0\}$  are attached to saliency seed set  $\mathcal{S}$ , i.e.,  $f(\mathbf{p}) = s_{\mathbf{p}}$  for all  $\mathbf{p} \in \mathcal{S}$ . Then  $f$  is a monotone submodular function with respect to  $\mathcal{S} \subset \mathcal{V}$ .*

Define  $L$  and  $\hat{L}$  as

$$L(\mathcal{S}) = \sum_{\mathbf{p} \in \mathcal{V}} f(\mathbf{p}; \mathcal{S}), \text{ and } \hat{L}(\mathcal{S}) = L(\mathcal{S}) - \sum_{\mathbf{p} \in \mathcal{S}} w(\mathbf{p}),$$

where  $f(\mathbf{p}; \mathcal{S})$  is the solution to LESD and  $w(\mathbf{p}) \geq 0$  is a function on  $\mathcal{F}_c$ . Then we have

**Corollary 2** *Both  $L(\mathcal{S})$  and  $\hat{L}(\mathcal{S})$  are submodular functions. Furthermore,  $L(\mathcal{S})$  is monotone with respect to  $\mathcal{S}$ .*

We also state some necessary definitions and lemmas<sup>1</sup>.

**Definition 3** *A set function  $f : 2^{\mathcal{V}} \rightarrow \mathbb{R}$  is monotone if for all subsets  $\mathcal{A} \subset \mathcal{B} \subset \mathcal{V}$ ,  $f(\mathcal{A}) \leq f(\mathcal{B})$ .*

<sup>1</sup>Please refer to [1] for all the definitions and lemmas in this material.

**Definition 4** *A set function  $f : 2^{\mathcal{V}} \rightarrow \mathbb{R}$  is submodular if the following inequality holds for all subsets  $\mathcal{A} \subset \mathcal{B} \subset \mathcal{V}$  and point  $\mathbf{q} \in \mathcal{V} \setminus \mathcal{B}$*

$$f(\mathcal{A} \cup \mathbf{q}) - f(\mathcal{A}) \geq f(\mathcal{B} \cup \mathbf{q}) - f(\mathcal{B}). \quad (3)$$

**Definition 5** *A set function  $f : 2^{\mathcal{V}} \rightarrow \mathbb{R}$  is modular if the following equality holds for any subsets  $\mathcal{A}, \mathcal{B} \subset \mathcal{V}$*

$$f(\mathcal{A}) + f(\mathcal{B}) = f(\mathcal{A} \cup \mathcal{B}) + f(\mathcal{A} \cap \mathcal{B}). \quad (4)$$

**Lemma 6** *Let  $f_1, f_2, \dots, f_n$  be submodular functions on  $\mathcal{V}$  and  $\alpha_1, \alpha_2, \dots, \alpha_n$  be non-negative constants. Then the function  $f = \sum_{i=1}^n \alpha_i f_i$  is submodular.*

**Lemma 7** *Let  $f_1$  and  $f_2$  be a submodular and a modular function on  $\mathcal{V}$ , respectively. Then the function  $f = f_1 - f_2$  is submodular.*

## 2. Proofs

**Proof** (of Theorem 1)<sup>2</sup> First, by energy conservation law in physics, the temperature of a diffusion system is always higher with more heat sources [2]. So we have that the visual attention score  $f$  is monotone with respect to the saliency seed set  $\mathcal{S}$ .

Then we use an inductive way to prove the submodularity of  $f$  with respect to  $\mathcal{S}$ . The proof consists of two steps: base and induction. Specifically, let  $d(\mathbf{p}, \mathbf{q})$  be the distance between  $\mathbf{p}$  and  $\mathbf{q}$ . Then we prove the submodularity of  $f$  by induction on  $d(\mathbf{p}, \mathbf{q})$ .

**Base Step:** For  $\mathbf{p}$  with  $d(\mathbf{p}, \mathbf{q}) = 0$  (i.e.,  $\mathbf{p} = \mathbf{q}$ ), we have  $f(\mathbf{p}; \mathcal{A} \cup \{\mathbf{q}\}) - f(\mathbf{p}; \mathcal{A}) \geq f(\mathbf{p}; \mathcal{B} \cup \{\mathbf{q}\}) - f(\mathbf{p}; \mathcal{B})$ . This is because  $f(\mathbf{p}; \mathcal{A} \cup \{\mathbf{q}\}) = f(\mathbf{p}; \mathcal{B} \cup \{\mathbf{q}\}) = s_{\mathbf{q}}$  and  $f(\mathbf{p}; \mathcal{A}) \leq f(\mathbf{p}; \mathcal{B})$  since  $f$  is monotone on  $\mathcal{V}$ .

**Induction Step:** Suppose inequality (3) holds for all  $\mathbf{p}$  with  $d(\mathbf{p}, \mathbf{q}) \leq r$  (for any  $r > 0$ ). We prove that (3) holds for all  $\mathbf{p}'$  with  $d(\mathbf{p}', \mathbf{q}) = r + \delta r$  in which  $\delta r > 0$  is a small

<sup>2</sup>This proof is only for discrete case and it is not difficult to draw the same conclusion for the continuous case.

108 perturbation. Specifically, the neighborhood set  $\mathcal{N}_{\mathbf{p}'}$  can be  
 109 separated into two subsets  $\mathcal{X} = \{\mathbf{x} | \mathbf{x} \in \mathcal{N}_{\mathbf{p}'}, d(\mathbf{x}, \mathbf{q}) \leq r\}$   
 110 and  $\mathcal{Y} = \{\mathbf{y} | \mathbf{y} \in \mathcal{N}_{\mathbf{p}'}, d(\mathbf{y}, \mathbf{q}) > r\}$ . Based on the induc-  
 111 tion hypotheses, we have (i)  $f(\mathbf{x}; \mathcal{A} \cup \{\mathbf{q}\}) - f(\mathbf{x}; \mathcal{A}) \geq$   
 112  $f(\mathbf{x}; \mathcal{B} \cup \{\mathbf{q}\}) - f(\mathbf{x}; \mathcal{B})$  for any  $\mathbf{x}$  in  $\mathcal{X}$  and (ii)  $f(\mathbf{y}; \mathcal{A} \cup$   
 113  $\mathbf{q}) = f(\mathbf{y}; \mathcal{A})$  and  $f(\mathbf{y}; \mathcal{B} \cup \mathbf{q}) = f(\mathbf{y}; \mathcal{B})$  for any  $\mathbf{y}$  in  $\mathcal{Y}$ . By  
 114 combining (i), (ii) and discrete formulation (2) together, we  
 115 have  $f(\mathbf{p}'; \mathcal{A} \cup \{\mathbf{q}\}) - f(\mathbf{p}'; \mathcal{A}) \geq f(\mathbf{p}'; \mathcal{B} \cup \{\mathbf{q}\}) - f(\mathbf{p}'; \mathcal{B})$ ,  
 116 which concludes the proof. 117

118 **Proof** (of Corollary 2) It is easy to check that  $W(\mathcal{S}) =$   
 119  $\sum_{\mathbf{p} \in \mathcal{S}} w(\mathbf{p})$  is a monotone function with respect to  $\mathcal{S}$ . Then  
 120 the conclusions in Corollary 2 can be directly proved by  
 121 Theorem 1, Lemma 6 and Lemma 7. 122

## 123 References 124

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