Supplemental Materials for Adaptive Partial Differential Equation Learning for Visual Saliency Detection

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1. General Statements

Before proving theoretical results, we first state our PDE system and its discretization as follows.

LESD:

$$\begin{aligned} &\mathrm{div}(\mathbf{K_p}\nabla f(\mathbf{p})) + \lambda(f(\mathbf{p}) - g(\mathbf{p})) = 0,\\ &s.t.\ f(\mathbf{g}) = 0,\ f(\mathbf{p}) = s_{\mathbf{p}},\ \mathbf{p}\in\mathcal{S}, \end{aligned} \tag{1}$$

where $\lambda \geq 0$.

Discretization:

$$f(\mathbf{p}) = \frac{1}{d_{\mathbf{p}} + \lambda}\left(\sum_{\mathbf{q}\in\mathcal{N}(\mathbf{p})} \mathbf{K_p}(\mathbf{q}) f(\mathbf{q}) + \lambda g(\mathbf{p})\right), \tag{2}$$

where $d_{\mathbf{p}} = \sum_{\mathbf{q}\in\mathcal{N}(\mathbf{p})} \mathbf{K_p}(\mathbf{q})$, $\mathbf{K_p}(\mathbf{q}) \geq 0$ and $g(\mathbf{p}) \geq 0$.

Then the main theoretical results (Theorem 1 in Section 2 and Corollary 2 in Section 3) in our manuscript are listed.

Theorem 1 *Let $f(\mathbf{p};\mathcal{S})$ be the visual attention score for image element $\mathbf{p}$ and the sources $\{s_{\mathbf{p}} \geq 0\}$ are attached to saliency seed set $\mathcal{S}$, i.e., $f(\mathbf{p}) = s_{\mathbf{p}}$ for all $\mathbf{p}\in\mathcal{S}$. Then f is a monotone submodular function with respect to $\mathcal{S}\subset\mathcal{V}$.*

Define L and $\hat{L}$ as

$$L(\mathcal{S}) = \sum_{\mathbf{p}\in\mathcal{V}} f(\mathbf{p};\mathcal{S}),\ \text{and}\ \hat{L}(\mathcal{S}) = L(\mathcal{S}) - \sum_{\mathbf{p}\in\mathcal{S}} w(\mathbf{p}),$$

where $f(\mathbf{p};\mathcal{S})$ is the solution to LESD and $w(\mathbf{p}) \geq 0$ is a function on $\mathcal{F}_c$. Then we have

Corollary 2 *Both $L(\mathcal{S})$ and $\hat{L}(\mathcal{S})$ are submodular functions. Furthermore, $L(\mathcal{S})$ is monotone with respect to $\mathcal{S}$.*

We also state some necessary definitions and lemmas[1].

Definition 3 *A set function $f : 2^{\mathcal{V}} \to \mathbb{R}$ is monotone if for all subsets $\mathcal{A}\subset\mathcal{B}\subset\mathcal{V}$, $f(\mathcal{A}) \leq f(\mathcal{B})$.*

[1]Please refer to [1] for all the definitions and lemmas in this material.

Definition 4 *A set function $f : 2^{\mathcal{V}} \to \mathbb{R}$ is submodular if the following inequality holds for all subsets $\mathcal{A}\subset\mathcal{B}\subset\mathcal{V}$ and point $\mathbf{q}\in\mathcal{V}/\mathcal{B}$*

$$f(\mathcal{A}\cup\mathbf{q}) - f(\mathcal{A}) \geq f(\mathcal{B}\cup\mathbf{q}) - f(\mathcal{B}). \tag{3}$$

Definition 5 *A set function $f : 2^{\mathcal{V}} \to \mathbb{R}$ is modular if the following equality holds for any subsets $\mathcal{A},\mathcal{B}\subset\mathcal{V}$*

$$f(\mathcal{A}) + f(\mathcal{B}) = f(\mathcal{A}\cup\mathcal{B}) + f(\mathcal{A}\cap\mathcal{B}). \tag{4}$$

Lemma 6 *Let $f_1, f_2, \cdots, f_n$ be submodular functions on $\mathcal{V}$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be non-negative constants. Then the function $f = \sum_{i=1}^{n} \alpha_i f_i$ is submodular.*

Lemma 7 *Let f_1 and f_2 be a submodular and a modular function on $\mathcal{V}$, respectively. Then the function $f = f_1 - f_2$ is submodular.*

2. Proofs

Proof (of Theorem 1) [2] First, by energy conservation law in physics, the temperature of a diffusion system is always higher with more heat sources [2]. So we have that the visual attention score f is monotone with respect to the saliency seed set $\mathcal{S}$.

Then we use a inductive way to prove the submodularity of f with respect to $\mathcal{S}$. The proof consists of two steps: base and induction. Specifically, let $d(\mathbf{p},\mathbf{q})$ be the distance between $\mathbf{p}$ and $\mathbf{q}$. Then we prove the submodularity of f by induction on $d(\mathbf{p},\mathbf{q})$.

Base Step: For $\mathbf{p}$ with $d(\mathbf{p},\mathbf{q}) = 0$ (i.e., $\mathbf{p} = \mathbf{q}$), we have $f(\mathbf{p};\mathcal{A}\cup\{\mathbf{q}\}) - f(\mathbf{p};\mathcal{A}) \geq f(\mathbf{p};\mathcal{B}\cup\{\mathbf{q}\}) - f(\mathbf{p};\mathcal{B})$. This is because $f(\mathbf{p};\mathcal{A}\cup\{\mathbf{q}\}) = f(\mathbf{p};\mathcal{B}\cup\{\mathbf{q}\}) = s_{\mathbf{q}}$ and $f(\mathbf{p};\mathcal{A}) \leq f(\mathbf{p};\mathcal{B})$ since f is monotone on $\mathcal{V}$.

Induction Step: Suppose inequality (3) holds for all $\mathbf{p}$ with $d(\mathbf{p},\mathbf{q}) \leq r$ (for any $r > 0$). We prove that (3) holds for all $\mathbf{p}'$ with $d(\mathbf{p}',\mathbf{q}) = r + \delta r$ in which $\delta r > 0$ is a small

[2]This proof is only for discrete case and it is not difficult to draw the same conclusion for the continuous case.

perturbation. Specifically, the neighborhood set $\mathcal{N}_{\mathbf{p}'}$ can be separated into two subsets $\mathcal{X} = \{\mathbf{x} | \mathbf{x} \in \mathcal{N}_{\mathbf{p}'}, d(\mathbf{x}, \mathbf{q}) \leq r\}$ and $\mathcal{Y} = \{\mathbf{y} | \mathbf{y} \in \mathcal{N}_{\mathbf{p}'}, d(\mathbf{y}, \mathbf{q}) > r\}$. Based on the induction hypotheses, we have (i) $f(\mathbf{x}; \mathcal{A} \cup \{\mathbf{q}\}) - f(\mathbf{x}; \mathcal{A}) \geq f(\mathbf{x}; \mathcal{B} \cup \{\mathbf{q}\}) - f(\mathbf{x}; \mathcal{B})$ for any $\mathbf{x}$ in $\mathcal{X}$ and (ii) $f(\mathbf{y}; \mathcal{A} \cup \mathbf{q}) = f(\mathbf{y}; \mathcal{A})$ and $f(\mathbf{y}; \mathcal{B} \cup \mathbf{q}) = f(\mathbf{y}; \mathcal{B})$ for any $\mathbf{y}$ in $\mathcal{Y}$. By combining (i), (ii) and discrete formulation (2) together, we have $f(\mathbf{p}'; \mathcal{A} \cup \{\mathbf{q}\}) - f(\mathbf{p}'; \mathcal{A}) \geq f(\mathbf{p}'; \mathcal{B} \cup \{\mathbf{q}\}) - f(\mathbf{p}'; \mathcal{B})$, which concludes the proof.

Proof (of Corollary 2) It is easy to check that $W(\mathcal{S}) = \sum_{\mathbf{p} \in \mathcal{S}} w(\mathbf{p})$ is a monotone function with respect to $\mathcal{S}$. Then the conclusions in Corollary 2 can be directly proved by Theorem 1, Lemma 6 and Lemma 7.

References

[1] S. Fujishige. *Submodular functions and optimization*, volume 58. Elsevier, 2005. 1

[2] J. Rice and B. Budiansky. Conservation laws and energy-release rates. *Journal of applied mechanics*, 40:201–203, 1973. 1