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Riemannian Manifold

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Synonyms

Differential manifold

Related Concepts

- ▶ Differential Geometry
- ▶ Manifold Learning
- ▶ Nonlinear Dimensionality Reduction

Definition

A *manifold* is a topological space that is locally Euclidean, indicating that near every point, there is a neighborhood that is topologically the same as the open unit ball in $\mathbb{R}^n$. A smooth manifold equipped with an inner product on each tangent space is called a *Riemannian manifold*, where various notions such as length, angles, areas (or volumes), curvature, and divergence of vector fields can be defined.

Relevance to Computer Vision

The need of using the manifold concept and Riemannian geometry arises naturally when considering the intrinsic properties of images. It is convenient to treat an image of size 64×64 as a data point in a 4096-dimensional vector space. Then we want to measure the distance or similarity between any two data points in this high-dimensional space. However, using the flat Euclidean structure in large scales seems to make little sense in this space. If you take two images that are very different, for example Arnold Schwarzenegger and Hillary Clinton, you cannot interpolate between them at all to get a facial image. (This example is from Kilian Q. Weinberger's homepage). One way is to impose some special metric in this space to make it curved, or equivalently we can transform these images into another feature space.

We can imagine an intuitive example of interpolating between digits 9 and 3. By directly doing linear interpolation in the original Euclidean space, we get blurry images. However, if following the curved manifolds of digits 9 and 3, we obtain a continuum of slowing morphing from 9 to 3. And this way, the interpolated images in the gap of two manifolds make much sense from the aspect of human eyes.

Furthermore, Riemannian geometry may play a role in the situations where a camera moves in a curved path (such as a circle) or a robot arm rotates along a given track.

K. Ikeuchi, *Computer Vision*,
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Mathematical Background

The concept of a manifold is central to many parts of geometry and modern mathematical physics, because it allows complicated structures to be described and understood in terms of the simpler local topological properties of Euclidean space. Manifolds arise naturally in a variety of mathematical and physical applications as "global objects." The study of manifolds combines many important areas of mathematics: it generalizes concepts such as curves and surfaces as well as ideas from linear algebra and topology.

In mathematics, a manifold is a topological space that locally resembles Euclidean space around each point. Precisely, each point of an n-dimensional manifold has a neighborhood that is homeomorphic to an open subset of Euclidean space $\mathbb{R}^n$. In other words, manifolds constitute a generalization of curves and surfaces into high-dimensional spaces. One-dimensional manifolds include lines and circles, but not figure eights "8" because they have crossing points that are not locally homeomorphic to Euclidean 1-space. Two-dimensional manifolds, also called surfaces, include the plane, the sphere, and the torus, which can all be embedded in three dimensional real space. On the other hand, the structure of three planes (e.g., the xy, yz, and zx planes in a xyz coordinate chart) intersected at the origin is an example of a non-manifold, since a neighborhood of any intersection point is neither 2D nor 3D.

Although a manifold locally resembles Euclidean space, globally it may not: manifolds in general are not homeomorphic to Euclidean space. An intuitive example is the **round/flat Earth problem**: the whole surface of the sphere is not homeomorphic to the Euclidean plane. The ancient belief is that the Earth was flat, as contrasted with the modern evidence that it is round. The discrepancy arises essentially from the fact that on the small scales, the Earth does indeed look flat, whereas on the large scales, the Earth surface is a sphere approximately. In general, any object that is nearly "flat" on small scales is a manifold. In a region it can be charted by means of map projections of the region into the Euclidean plane; in the context of manifolds, they are called *charts*. More concisely, any object that can be "charted" is a manifold.

When a region appears in two neighboring charts, the two representations may not coincide exactly, and a transformation is needed to pass from one chart to the other, called a *transition map*. *Smooth manifolds* (also called differentiable manifolds) are manifolds for which overlapping charts "relate smoothly" to each other, meaning that the inverse of one followed by the other is an infinitely differentiable map from Euclidean space to itself. Such two charts are called "compatible." A differentiable manifold is a type of manifold that is locally similar enough to a linear space to allow one to do calculus. One may then apply calculus while working within the individual charts, since each chart lies within a linear space to which the usual rules of calculus apply. If the charts are suitably compatible (viz., the transition from one chart to another is differentiable), then computations done in one chart are valid in any other differentiable chart. With this differentiable structure equipped on manifolds, we can define the globally differentiable tangent space, differentiable functions, and differentiable tensor and vector fields.

Differentiable manifolds are very important in physics. Special kinds of differentiable manifolds form the basis for physical theories such as classical mechanics, general relativity, and the Yang-Mills theory. It is possible to develop a calculus for differentiable manifolds, and the study of calculus on differentiable manifolds is known as *differential geometry*.

A Riemannian metric on a differentiable manifold allows distances and angles to be measured. A "Riemannian manifold" is a differentiable manifold in which each tangent space is equipped with an inner product $\langle\cdot,\cdot\rangle$ in a manner which varies smoothly from point to point. Given two tangent vectors u and v, the inner product $\langle u, v\rangle$ gives a real number. This allows one to define various notions such as length, angles, areas (or volumes), curvature, and divergence of vector fields.

Carl Friedrich Gauss (1777–1855) may have been the first to consider abstract spaces as mathematical objects in their own right. His Theorema Egregium (Remarkable Theorem or Totally Awesome Theorem) gives a method for computing the curvature of a surface without considering the ambient space in which the surface lies. In modern terms, the theorem proved that the Gaussian curvature $K = \kappa_1\kappa_2$ of the surface is an intrinsic property, where κ_1 and κ_2 are principal curvatures at one point. Since then, manifold theory has come to focus exclusively on these intrinsic properties (or invariants), while largely ignoring the extrinsic properties of the ambient space.

Bernhard Riemann (1826–1866) was the first one to do extensive work generalizing the idea of intrinsic geometry to higher dimensions. The name manifold comes from Riemann's original German term, Mannigfaltigkeit, which William Kingdon Clifford translated as "manifoldness." Riemann developed his theory of higher dimensions and delivered his inaugural lecture at Göttingen in 1854 entitled "Ueber die Hypothesen welche der Geometrie zu Grunde liegen" ("On the hypotheses which underlie geometry") [1]. Riemann found the correct way to extend to n dimensions the differential geometry of surfaces, which Gauss (as Riemann's tutor) proved in his Theorema Egregium. This lecture founded the field of Riemannian geometry.

Gauss highly appreciated Riemann's lecture "which surpassed all his expectations, in the greatest astonishment, and on the way back from the faculty meeting he spoke to Wilhelm Weber, with the greatest appreciation, and with an excitement rare for him, about the depth of the ideas presented by Riemann." [2, Vol.2, pp.134]

Riemann's inaugural lecture was only published 12 years later in 1868 by Dedekind, 2 years after his death. Its early reception appears to have been slow. But it is now recognized as one of the most important works in geometry and specifically set the stage for Albert Einstein's general theory of relativity (published in 1916). Einstein used the theory of Riemannian manifolds (formally pseudo-Riemannian manifolds) to develop his general theory of relativity. General relativity generalizes special relativity and Newton's law of universal gravitation, treating the gravity of space and time (or spacetime) as the curvature of a Riemannian space. In particular, his equations for gravitation are constraints on the curvature of space-time. As Riemann laid the foundations of the mathematics of general relativity, Riemannian geometry had received extensive attention in mathematics and theoretical physics from that time.

For more material on the background, one can refer to https://en.wikipedia.org/wiki/Manifold and https://en.wikipedia.org/wiki/Riemannian_manifold.

Theory

In the following we present formal definitions and several elementary results [3].

Definition 1 A *manifold* M of dimension d is a connected paracompact Hausdorff space for which every point has a neighborhood U that is homeomorphic to an open subset Ω of $\mathbb{R}^d$. Such a homeomorphism $x : U \mapsto \Omega$ is called a *(coordinate) chart*. An *atlas* is a family $\{U_\alpha, x_\alpha\}$ of charts for which the U_α constitute an open covering of M.

Definition 2 An atlas $\{U_\alpha, x_\alpha\}$ on a manifold is called *differentiable* if all chart transitions

$$x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta) \mapsto x_\beta(U_\alpha \cap U_\beta)$$

are differentiable of class C^∞ (in case $U_\alpha \cap U_\beta \neq \emptyset$). A maximal differentiable atlas is called a differentiable structure, and a *differentiable manifold* of dimension d is a manifold of dimension d with a differentiable structure.

The next key concept is called "tangent space," which is a generalization of the tangent plane of a 2D surface. Suppose that a regular surface S is represented by $\mathbf{r}(u, v) = \{x(u, v), y(u, v), z(u, v)\}$, where $(u, v) \in \mathbb{R}^2$ and $(x, y, z) \in \mathbb{R}^3$. Thus as a main part of the functional increment, the differential

$$d\mathbf{r}(u, v) = \mathbf{r}_u(u, v)du + \mathbf{r}_v(u, v)dv$$

gives a tangent vector with (du, dv) as coordinates under the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ (generally not orthonormal). This vector space formed by basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ is called a tangent space of surface S at point p, denoted by T_pS.

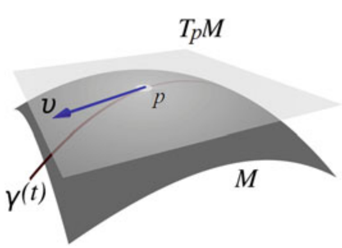
Definition 3 Let p be a point in an d-dimensional differentiable manifold M, and attach at p a copy of $\mathbb{R}^d$ tangential to M. The resulting structure is called the tangent space of M at p, denoted by T_pM. If γ is a smooth curve passing through p, then the derivative of γ at p (also known as the velocity of the curve) is a tangent vector in T_pM.

See Fig. 1 for an illustrative example. Other definitions of the tangent space are possible. For example, a tangent vector at p may be defined as directional derivatives X of smooth functions on M that satisfies the Leibniz rule

$$X(f \cdot g)(p) = X(f(p))g(p) + f(p)X(g(p)),$$

which does not involve local coordinates.

Definition 4 Let M be a differentiable manifold of dimension d and $p \in M$. A *Riemannian metric* on M is a family of (positive definite) inner products $g_p\colon T_pM \times T_pM \mapsto \mathbb{R}$ such that for all differentiable vector fields X, Y on M, $p \mapsto g_p(X(p), Y(p))$ defines a smooth function $M \mapsto \mathbb{R}$. In other words, a Riemannian metric g is a symmetric (0,2)-tensor that is positive definite (i.e., $g(X, X) > 0$ for all tangent vectors $X \neq 0$). A *Riemannian manifold* is a differentiable manifold equipped with a Riemannian metric.



Riemannian Manifold, Fig. 1 The tangent space

The inner product of two tangent vector $X, Y \in T_pM$ with coordinate representations $X = \sum_i X^i \frac{\partial}{\partial x^i}$, $Y = \sum_j Y^j \frac{\partial}{\partial x^j}$ then is

$$\langle X, Y\rangle = g_p(X, Y) = g_{ij}(p)X^iY^j.$$

In particular $\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle = g_{ij}$. Formally, the metric tensor can be written in terms of the dual basis $\{dx^1, \ldots, dx^d\}$ of the cotangent space as

$$g = \sum_{i,j} g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j.$$

For a smooth parametrized curve $\gamma : [a, b] \mapsto M$, its *length* is defined as

$$\begin{aligned}
L(\gamma) &= \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt \\
&= \int_a^b \sqrt{\left\langle \frac{d\gamma}{dt}(t), \frac{d\gamma}{dt}(t) \right\rangle} dt \\
&= \int_a^b \sqrt{\left\langle \sum_i \frac{\partial\gamma}{\partial x^i}\frac{dx^i}{dt}, \sum_j \frac{\partial\gamma}{\partial x^j}\frac{dx^j}{dt} \right\rangle} dt \\
&= \int_a^b \sqrt{\sum_{i,j} g\left(\frac{\partial\gamma}{\partial x^i}, \frac{\partial\gamma}{\partial x^j}\right)\frac{dx^i}{dt}\frac{dx^j}{dt}} dt \\
&= \int_a^b \sqrt{\sum_{i,j} g_{ij}(\gamma(t))\frac{dx^i}{dt}\frac{dx^j}{dt}} dt,
\end{aligned}$$

which is exactly the original starting point of Riemann.

Similarly, a *volume element* can be expressed by

$$dV = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n,$$

where $\det(g_{ij})$ is the determinant of the matrix representation of the metric tensor and $\{dx^i\}$ is the dual coframe. Hence the volume of an oriented manifold M is defined to be

$$\int_M dV = \int_M \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n.$$

The following theorem provides a guarantee of the existence of the Riemannian metric.

Theorem 1 *Each differentiable manifold may be equipped with a Riemannian metric.*

One can read [4] for more details on Riemannian geometry and [3] for connections to geometric analysis. For those not familiar with differential geometry of curves and surfaces, please refer to [5] (without tensor analysis).

Application

Riemannian normal coordinates, introduced by Riemann in his inaugural lecture in 1854, have been used to manifold learning. Local coordinate charts can be constructed to embed data points from a high-dimensional ambient space into a low-dimensional feature space [6].

Riemannian manifolds have found successful applications for video representations in visual classification tasks, since a discriminant Riemannian metric can encode the nonlinear geometry of the underlying Riemannian manifolds. In [7] a metric learning framework was presented to learn a distance metric across a Euclidean space and a Riemannian manifold to fuse the average appearance and pattern variation of faces within one video.

Visual data often forms a special manifold structure lying on a lower dimensional space. An efficient clustering method on Riemannian manifolds was proposed in [8], and experiments over several image and video datasets demonstrated the favorable computational complexity of the proposed clustering algorithm.

In video analysis and more generally activity recognition, temporal evolutions of features can be viewed as trajectories on Riemannian manifolds. A transported square-root vector field [9] on Riemannian manifolds was used to model these trajectories, which successfully applied to visual speech recognition.

Grassmann manifold, as a special case of general Riemannian manifolds, has attracted much interests in the community of computer vision. A robust estimation approach based on Grassmann manifolds [10] was employed for chromatic noise filtering, fundamental matrix estimation, planar homography, and affine motion factorization.

The monograph [11] presents a comprehensive treatise on Riemannian geometric computations and related statistical inferences in several computer vision problems, including face recognition, activity recognition, object detection, biomedical image analysis, and structure from motion.

Recently, Riemannian geometry has been applied to the study of deep neural networks [12]. They found that neural networks are learning systems of differential equations governing the coordinate transformations that represent the data manifold. Also a closed form solution of the metric tensor on the underlying data manifold can be found by back-propagating the coordinate representations learned by neural networks.

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