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Matrix factorizations for reversible integer implementation of orthonormal M-band wavelet transforms[☆]

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Abstract

This paper presents a matrix factorization method for implementing orthonormal M-band wavelet reversible integer transforms. Based on an algebraic construction approach, the polyphase matrix of orthonormal M-band wavelet transforms can be factorized into a finite sequence of elementary reversible matrices that map integers to integers reversibly. These elementary reversible matrices can be further factorized into lifting matrices, thus we extend the classical lifting scheme to a more flexible framework.

Keywords: M-band wavelets; Matrix factorizations; Reversible integer transforms; The lifting scheme

1. Introduction

For signal processing and image compression, M-band wavelets have attracted considerable attention due to their ability to provide much more freedom than the classical two-band wavelets, such as the coexistence of orthogonality and linear phase [1,2]. However, the increased degrees of freedom make it challenging to construct general M-band wavelets, and also difficult to efficiently implement M-band wavelet transforms in the lifting scheme or other fast algorithms.

As there are $M-1$ high-pass wavelet filters and only one low-pass scaling filter in an M-band wavelet system, commonly a two-step construction procedure is applied to reduce the design difficulties. The first step is to design the scaling filter with K-regularity [3,4], linear-phase [5], and other properties [6,7]. Then in the second step, wavelet filters are chosen to meet some special requirements with the given scaling filter [5,3,8,9,4,10,11]. Since the scaling filter and the wavelet filters are designed separately and often only specific scaling filters are studied, the two-step construction may not completely exploit the freedom provided by the general M-band wavelets. The lifting scheme [12–14] offers a new approach to design and efficiently implement the classical two-band wavelet transforms. In addition, wavelet transforms can be further implemented with reversible integer mapping [15,16],

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which is essential for lossless source coding by transformation. In [17–19], the lifting scheme is directly generalized to M-band wavelets. In [20–22], the polyphase transfer matrix for M-band wavelets is first represented by the lattice structure or the LU factorization, then is further decomposed into lifting steps or ladder structures.

In [23] an algebraic method of solving constraint equations is presented for construction of orthonormal M-band wavelets. The solution is obtained via matrix decomposition. It is natural to factorize the construction matrices further into lifting steps, or into elementary reversible matrices that directly map integers to integers, which is proposed in this paper. We will be concerned with real filter banks throughout the paper.

The rest of this paper is organized as follows. In Section 2 we review some basic facts of the algebraic construction method [23], and reversible integer transforms [15,16,24]. Section 3 describes the main results of our factorization for reversible integer transforms, and in Section 4 we discuss the relations with the lifting scheme. The paper is concluded in Section 5.

2. Preliminary

In this section we recall some basic concepts and results, which will be used in the subsequent paper.

2.1. M-band wavelets

Suppose the filter bank matrix [1,25] of M-band wavelets with length ML is $A=[A_0,A_1,\ldots,A_{L-1}]$, where A_j are $M\times M$ matrices with $M\geqslant 2$ and $L\geqslant 2$. L is often called the overlapping factor [21,20]. The first row of A is for the low-pass filter, and other $M-1$ rows are for high-pass filters of the wavelets. Thus, the polyphase matrix [14,26] is

$$P(z)=A_0+A_1z^{-1}+\cdots+A_{L-1}z^{-(L-1)}. \tag{2.1}$$

The constraint conditions for an orthonormal (OR) M-band filter bank with perfect reconstruction (PR) property are as follows:

$$\begin{cases} \hat{S}e=\sqrt{M}e_1 & \text{(low-pass and high-pass)},\\ PP^{\mathrm{T}}=I & \text{(OR)},\\ QQ^{\mathrm{T}}=I & \text{(PR)}, \end{cases} \tag{2.2}$$

where

$$\hat{S}=\sum_{j=0}^{L-1}A_j,\quad e=(1,1,\ldots,1)^{\mathrm{T}},\quad e_1=(1,0,\ldots,0)^{\mathrm{T}}, \tag{2.3}$$

$$P=\begin{bmatrix} A_0 & A_1 & \cdots & A_{L-1} & & & \\ & A_0 & A_1 & \cdots & A_{L-1} & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & A_0 & A_1 & \cdots & A_{L-1} \end{bmatrix}, \tag{2.4}$$

$$Q=\begin{bmatrix} A_0^{\mathrm{T}} & A_1^{\mathrm{T}} & \cdots & A_{L-1}^{\mathrm{T}} & & & \\ & A_0^{\mathrm{T}} & A_1^{\mathrm{T}} & \cdots & A_{L-1}^{\mathrm{T}} & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & A_0^{\mathrm{T}} & A_1^{\mathrm{T}} & \cdots & A_{L-1}^{\mathrm{T}} \end{bmatrix}. \tag{2.5}$$

For the cases of $L=2$ and $L=3$, the following results have been proved in [23]:

- For the case of $L=2$, $A=[A_0,\ A_1]$ satisfy (2.2) if and only if they have the following decompositions:

$$A_0=U\begin{bmatrix} I_{n_0} & 0\\ 0 & 0\end{bmatrix}V^{\mathrm{T}},\quad A_1=U\begin{bmatrix} 0 & 0\\ 0 & I_{n_1}\end{bmatrix}V^{\mathrm{T}}, \tag{2.6}$$

where $n_0+n_1=M$, and U and V are orthogonal matrices with $UV^{\mathrm{T}}e=\sqrt{M}e_1$.
- For the case of $L=3$, $A=[A_0,\ A_1,\ A_2]$ satisfy (2.2) if and only if they have the decompositions $A_k=US_kV^{\mathrm{T}},\ k=0,1,2$, where

$$S_0=\mathrm{diag}(S,I_{n_0},0,0,0),$$
$$S_1=\begin{bmatrix} 0 & 0 & 0 & 0 & -C\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I_{n_1} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ C & 0 & 0 & 0 & 0\end{bmatrix},$$
$$S_2=\mathrm{diag}(0,0,0,I_{n_2},S),\quad S=\mathrm{diag}(s_1,s_2,\ldots,s_r),$$
$$C=\mathrm{diag}(c_1,c_2,\ldots,c_r),\quad 0<s_i,\ c_i<1,$$
$$c_i^2+s_i^2=1,\quad 2r+n_0+n_1+n_2=M, \tag{2.7}$$

and U and V are orthogonal matrices and satisfy $U(S_0+S_1+S_2)V^{\mathrm{T}}e=\sqrt{M}e_1$.

2.2. Reversible integer transform

An *integer factor* is defined as ± 1 for real numbers. A *triangular elementary reversible matrix* (TERM) is an upper or a lower triangular square matrix with integer factor diagonal entries, and a *single-row elementary reversible matrix* (SERM) is a square matrix with integer factor diagonal entries and only one row of off-diagonal entries that are not all zeros. If all the diagonal entries are equal to 1, the matrix is called a unit TERM or a unit SERM.

One of the important properties of elementary reversible matrices is that we can use reversible integer transforms to approximate them. For example, let $U = [u_{ij}]$ be an $M \times M$ upper TERM, the linear transform $y = Ux$ can be approximated by the following reversible integer transform:

$$\begin{cases} y_i = u_{ii}x_i + [\sum_{j=i+1}^{M} u_{ij}x_j], & i = 1, 2, \ldots, M-1, \\ y_M = u_{MM}x_M, \end{cases}$$

where $[r]$ denotes the rounding arithmetic for any real number r. Because u_{ii} is an integer factor that does not change the magnitude, the output y_i are integers if the input x_i are integers. Moreover, x_i can be recovered from y_i in the order of $x_M, x_{M-1}, \ldots, x_1$.

The following result shows that the normalized matrices can be factorized into TERMs or SERMs, which has been proved in [16].

Lemma 2.1. *If an $M \times M$ matrix A satisfies* $\det(A) = \pm 1$, *then A has*

- *a TERM factorization: $A = PLU\,S_0$, where P is a permutation matrix, L a unit lower TERM, U a unit upper TERM, and S_0 a SERM with nonzero off-diagonal entries in the bottom row,*
- *a SERM factorization: $A = PS_M S_{M-1} \cdots S_1 S_0$, where P is a permutation matrix, S_m ($m = M, M-1, \ldots, 1$) unit SERMs with nonzero off-diagonal entries in the mth row, and S_0 a SERM with nonzero off-diagonal entries in the bottom row.*

3. Factorizations

In this section, we give the TERM and the SERM factorization of the polyphase matrix $P(z)$ of an orthonormal M-band filter bank $A = [A_0, A_1, \ldots, A_{L-1}]$ for the cases of $L = 2$ and $L = 3$.

3.1. The case of $L = 2$

For the case of $L = 2$, by (2.6), the polyphase matrix has the following form:

$$P(z) = A_0 + A_1 z^{-1} = U \begin{bmatrix} I_{n_0} & 0 \\ 0 & I_{M-n_0} z^{-1} \end{bmatrix} V^{\mathrm{T}}.$$

Because U and V are both orthonormal matrices, $\det(U) = \pm 1$, $\det(V) = \pm 1$. By Lemma 2.1, U and V have TERM factorization of form $PLU\,S_0$ and SERM factorization of form $PS_M S_{M-1} \cdots S_1 S_0$. The intermediate matrix

$$\begin{bmatrix} I_{n_0} & 0 \\ 0 & I_{M-n_0} z^{-1} \end{bmatrix}$$

is equivalent to identity matrix, except for a translation of the input signal corresponding to the lower-right part. Thus, reversible integer transforms can be implemented for M-band wavelets of the case $L = 2$.

Example 3.1 (*2-Regular 3-band wavelets*). For $L = 2$ and $M = 3$, the regularity constraints can be imposed to get the following filter bank [4,3,2,23] as shown in Table 1.

For this filter bank the associated decompositions as in (2.6) are as follows:

$$A_0 = \begin{bmatrix} 0.3384 & 0.5308 & 0.7233 \\ -0.1174 & 0.5443 & -0.0187 \\ 0.4036 & -0.6285 & 0.4606 \end{bmatrix} = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^{\mathrm{T}},$$

Table 1
2-Regular 3-band filter bank

$h_{0,n}$	$h_{1,n}$	$h_{2,n}$
0.33838609728386	−0.11737701613483	0.40363686892892
0.53083618701374	0.54433105395181	−0.62853936105471
0.72328627674361	−0.01870574735313	0.46060475252131
0.23896417190576	−0.69911956479289	−0.40363686892892
0.04651408217589	−0.13608276348796	−0.07856742013185
−0.14593600755399	0.42695403781698	0.24650202866523

$$A_1 = \begin{bmatrix} 0.2390 & 0.0465 & -0.1459 \\ -0.6991 & -0.1361 & 0.4270 \\ -0.4036 & -0.0786 & 0.2465 \end{bmatrix} = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{\mathrm{T}},$$

where

$$U = \begin{bmatrix} -0.2838 & -0.8475 & 0.4485 \\ 0.8304 & -0.4511 & -0.3270 \\ 0.4794 & 0.2796 & 0.8318 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.8419 & -0.1210 & 0.5259 \\ -0.1639 & -0.8712 & -0.4628 \\ 0.5142 & -0.4758 & 0.7136 \end{bmatrix}.$$

It is easy to verify that U has the TERM factorization: $U = L_0 U_0 S_0$, where

$$L_0 = \begin{bmatrix} 1 & & \\ -0.1057 & 1 & \\ 2.8608 & 5.1901 & 1 \end{bmatrix},$$

$$U_0 = \begin{bmatrix} 1 & -3.3187 & 0.4485 \\ & 1 & -0.2796 \\ & & 1 \end{bmatrix},$$

$$S_0 = \begin{bmatrix} 1 & & \\ & & 1 \\ -2.8628 & 5.5105 & 1 \end{bmatrix}.$$

We can also factorize U into SERMs by factorizing $L_0 U_0$ into three SERMs, that is, $L_0 U_0 = S_3 S_2 S_1$, so we have the SERM factorization: $U = S_3 S_2 S_1 S_0$, where

$$S_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ 3.4094 & 5.1901 & 1 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 1 & & \\ -0.1057 & 1 & -0.2796 \\ & & 1 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 1 & -3.3187 & 0.4485 \\ & 1 & \\ & & 1 \end{bmatrix}.$$

Similarly, we have $V = \widetilde{L}_0 \widetilde{U}_0 \widetilde{S}_0$, where

$$\widetilde{L}_0 = \begin{bmatrix} 1 & & \\ -1.7847 & 1 & \\ 3.0136 & -3.9328 & 1 \end{bmatrix},$$

$$\widetilde{U}_0 = \begin{bmatrix} 1 & 2.1859 & 0.5259 \\ & 1 & 0.4758 \\ & & 1 \end{bmatrix},$$

$$\widetilde{S}_0 = \begin{bmatrix} 1 & & \\ & 1 & \\ -3.5024 & -4.3866 & 1 \end{bmatrix}.$$

Of course, we may also give the SERM decomposition of V.

In summary, we have $P(z) = L_0 U_0 S_0 \operatorname{diag}(z^{-1}, 1, 1)(\widetilde{L}_0 \widetilde{U}_0 \widetilde{S}_0)^{\mathrm{T}}$, which is a reversible integer factorization of $P(z)$.

3.2. The case of $L = 3$

For the case of $L = 3$, by (2.7), the polyphase matrix has the following form:

$$P(z) = A_0 + A_1 z^{-1} + A_2 z^{-2} = U \begin{bmatrix} S & & -Cz^{-1} \\ & B & \\ Cz^{-1} & & Sz^{-2} \end{bmatrix} V^{\mathrm{T}},$$

where

$$B = \begin{bmatrix} I_{n_0} & & \\ & I_{n_1} z^{-1} & \\ & & I_{n_2} z^{-2} \end{bmatrix}.$$

Noting that

$$M = \begin{bmatrix} S & & -Cz^{-1} \\ & B & \\ Cz^{-1} & & Sz^{-2} \end{bmatrix} = \begin{bmatrix} S & & -Cz^{-1} \\ & I & \\ Cz^{-1} & & Sz^{-2} \end{bmatrix} \begin{bmatrix} I & & \\ & B & \\ & & I \end{bmatrix}$$

and the transform with B can be implemented for reversible integer mapping directly, we only need to consider how to factorize the matrix

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix}$$

into lifting matrices. Noting that C, S, and $C+S$ are all nonsingular, $S^2+C^2=I$, and $SC=CS$, and using the following useful equalities [14,16,27]:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & A_{12} \\ A_{21}+A_{22}A_{12}^{-1}(I-A_{11}) & A_{22} \end{bmatrix} \times \begin{bmatrix} I & 0 \\ -A_{12}^{-1}(I-A_{11}) & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \times \begin{bmatrix} A_{11} & A_{12} \\ 0 & I \end{bmatrix},$$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{21} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix},$$

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix},$$

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & ACB^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

$$\begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\alpha}-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ \alpha-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{\alpha} \\ 0 & 1 \end{bmatrix}, = \begin{bmatrix} 1 & 0 \\ \frac{1}{\alpha} & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha-1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\alpha}-1 \\ 0 & 1 \end{bmatrix},$$

we can obtain many different reversible integer factorizations. For the limitation of the paper length, we here just present four of them as follows.

1. The factorization with three TERMs:

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix} = \begin{bmatrix} I & -Cz^{-1} \\ (I-S)C^{-1}z^{-1} & Sz^{-2} \end{bmatrix} \begin{bmatrix} I & 0 \\ C^{-1}(I-S)z & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I-S)C^{-1}z^{-1} & Iz^{-2} \end{bmatrix} \begin{bmatrix} I & -Cz^{-1} \\ 0 & I \end{bmatrix} \times \begin{bmatrix} I & 0 \\ C^{-1}(I-S)z & I \end{bmatrix}.$$

2. The factorization with three TERMs and one permutation matrix:

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix} = \begin{bmatrix} -Cz^{-1} & S \\ Sz^{-2} & Cz^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & S \\ (S+CS^{-1}C)z^{-2}+CS^{-1}z^{-1} & Cz^{-1} \end{bmatrix} \times \begin{bmatrix} I & 0 \\ -S^{-1}(I+Cz^{-1}) & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ S^{-1}z^{-2}+CS^{-1}z^{-1} & -Iz^{-2} \end{bmatrix} \begin{bmatrix} I & S \\ 0 & I \end{bmatrix} \times \begin{bmatrix} I & 0 \\ -S^{-1}(I+Cz^{-1}) & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

3. The factorization with four TERMs:

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & Iz^{-1} \end{bmatrix} \begin{bmatrix} S-C & -(C+S)z^{-1} \\ C & Sz^{-1} \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & Iz^{-1} \end{bmatrix} \begin{bmatrix} I & -(C+S) \\ X & S \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & Iz^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I & I \\ 0 & Iz^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} I & -(C+S) \\ 0 & I \end{bmatrix} \times \begin{bmatrix} I & 0 \\ Y & Iz^{-1} \end{bmatrix},$$

where

$$X = C - S(C+S)^{-1}(I - S + C),$$
$$Y = (C+S)^{-1}(I - S + C).$$

4. The factorization with seven TERMs:

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ CS^{-1}z^{-1} & S^{-1}z^{-2} \end{bmatrix} \begin{bmatrix} S & -Cz^{-1} \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ CS^{-1}z^{-1} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S^{-1}z^{-2} \end{bmatrix} \begin{bmatrix} I & -Cz^{-1} \\ 0 & I \end{bmatrix} \times \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ CS^{-1}z^{-1} & I \end{bmatrix} \begin{bmatrix} I & -CSz \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S^{-1}z^{-2} \end{bmatrix} \times \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ CS^{-1}z^{-1} & I \end{bmatrix} \begin{bmatrix} I & -CSz \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Iz^{-2} \end{bmatrix} \times \begin{bmatrix} S & 0 \\ 0 & S^{-1} \end{bmatrix},$$

where

$$\begin{bmatrix} S & 0 \\ 0 & S^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ S^{-1} - I & I \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ S - I & I \end{bmatrix} \times \begin{bmatrix} I & -S^{-1} \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ -S^{-1} & I \end{bmatrix} \begin{bmatrix} I & S - I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \times \begin{bmatrix} I & S^{-1} - I \\ 0 & I \end{bmatrix}.$$

Thus, reversible integer transforms can be implemented for M-band wavelets of the case $L = 3$.

Example 3.2 (*3-Regular 3-band wavelets*). The example filter bank of 3-regular 3-band in [2–4,23] can be given by the following decompositions:

$$A_0 = \begin{bmatrix} 0.2031 & 0.4232 & 0.7073 \\ -0.4214 & 0.7247 & 0.0395 \\ 0.0284 & -0.1610 & -0.1081 \end{bmatrix} = U \begin{bmatrix} 0.6807 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{\mathrm{T}},$$

$$A_1 = \begin{bmatrix} 0.4462 & 0.1986 & -0.1772 \\ -0.3270 & -0.2744 & 0.3241 \\ 0.4675 & -0.2401 & 0.4897 \end{bmatrix} = U \begin{bmatrix} 0 & 0 & -0.7326 \\ 0 & 0 & 0 \\ 0.7326 & 0 & 0 \end{bmatrix} V^{\mathrm{T}},$$

$$A_2 = \begin{bmatrix} -0.0720 & -0.0444 & 0.0473 \\ -0.0681 & -0.0420 & 0.0447 \\ -0.4959 & -0.3061 & 0.3255 \end{bmatrix} = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.6807 \end{bmatrix} V^{\mathrm{T}},$$

where

$$U = \begin{bmatrix} -0.6948 & 0.7049 & -0.1424 \\ 0.7192 & 0.6817 & -0.1346 \\ 0.0022 & 0.1960 & -0.9806 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.6525 & -0.1496 & 0.7429 \\ 0.3332 & 0.8238 & 0.4585 \\ -0.6806 & 0.5467 & -0.4877 \end{bmatrix}.$$

It is easy to verify that $U = L_0 U_0 S_0$ and $V = \widetilde{L}_0 \widetilde{U}_0 \widetilde{S}_0$, where

$$L_0 = \begin{bmatrix} 1 & & \\ 2.3217 & 1 & \\ 11.6735 & -1.6244 & 1 \end{bmatrix},$$

$$\widetilde{L}_0 = \begin{bmatrix} 1 & & \\ 1.3531 & 1 & \\ -1.7654 & 0.3222 & 1 \end{bmatrix},$$

$$U_0 = \begin{bmatrix} 1 & 0.7157 & -0.1424 \\ & 1 & 0.1960 \\ & & 1 \end{bmatrix},$$

$$\widetilde{U}_0 = \begin{bmatrix} 1 & -0.1139 & 0.7429 \\ & 1 & -0.5467 \\ & & 1 \end{bmatrix},$$

$$S_0 = \begin{bmatrix} 1 & & \\ & 1 & \\ 11.9021 & -9.9760 & 1 \end{bmatrix},$$

$$\widetilde{S}_0 = \begin{bmatrix} 1 & & \\ & 1 & \\ -2.2243 & -0.0481 & 1 \end{bmatrix}.$$

The middle matrix M can be factorized into three SERMs: $M = S_3 S_1 \widetilde{S}_3$, where

$$M = \begin{bmatrix} 0.6807 & & -0.7326z^{-1} \\ & 1 & \\ 0.7326z^{-1} & & 0.6807z^{-2} \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ 0.4359z^{-1} & 0 & z^{-2} \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 1 & 0 & -0.7326z^{-1} \\ & 1 & \\ & & 1 \end{bmatrix},$$

$$\widetilde{S}_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ 0.4359z & 0 & 1 \end{bmatrix}.$$

In summary, we have $P(z) = L_0 U_0 S_0 S_3 S_1 \widetilde{S}_3 (\widetilde{L}_0 \widetilde{U}_0 \widetilde{S}_0)^{\mathrm{T}}$, which is a reversible integer factorization of $P(z)$.

4. Relations with the lifting matrices

It has been proved in [14,19] that any biorthogonal two-band or M-band wavelet transform can be obtained using the lifting scheme [12,13]. This corresponds to a factorization of the polyphase matrix into a sequence of lifting matrices, dual lifting matrices, and one diagonal scaling matrix. For the two-band [12–14] or the M-band cases [18,19], a lifting matrix is a lower triangular square matrix whose diagonal elements are 1 and nonzero nondiagonal elements are only in the first column, which corresponds to modify the $M-1$ high-pass filters as the low-pass filter is fixed. A dual lifting matrix is an upper triangular square matrix whose diagonal elements are 1 and nonzero nondiagonal elements are only in the first row, which corresponds to modify the low-pass filter based on the high-pass filters. In [28,29], a lifting matrix is defined as a matrix whose diagonal elements are 1, and only one nondiagonal element is nonzero. Generally, the nonzero nondiagonal elements of a lifting matrix can be located in any row or column other than the first.

Here we show that a lifting matrix (or a dual lifting matrix), whose nonzero nondiagonal elements are strictly located in the first column (or in the first row), can be relaxed to be any *elementary reversible matrix* such as a TERM or a SERM. Obviously, a lifting matrix is a TERM or a SERM. On the other hand, a TERM can be converted into a sequence of SERMs by extracting each row sequentially [16]. A SERM can be directly factorized into a sequence of lifting matrices, or be converted into a SERM corresponding to the first row and two permutation matrices. A permutation matrix can also be converted into lifting matrices with nonzero nondiagonal elements in the first row or the first column. The essence can be conveyed by the following simple examples. Let

$$S = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ a & b & 1 & c & d \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}.$$

Straightforward calculations verify that

$$S = L_1 U_1 L_2 U_2 L_3, \quad S = P_{13} S_1 P_{13}, \quad P = D(\widetilde{L}_1 \widetilde{U}_1)^2,$$

where

$$L_1 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ 1 & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 1 & b & & c & d \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ -1 & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix},$$

$$U_2 = \begin{bmatrix} 1 & -b & & -c & -d \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ a & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \quad P_{13} = \begin{bmatrix} 0 & & 1 & & \\ & 1 & & & \\ 1 & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 1 & a & b & c & d \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

$$\widetilde{L}_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ -1 & & & 1 \end{bmatrix}, \quad \widetilde{U}_1 = \begin{bmatrix} 1 & & -1 & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

5. Conclusion

In this paper, we first review an algebraic method for construction of orthonormal M-band wavelets and reversible integer transforms. Based on the algebraic construction, the polyphase matrix of the M-band wavelet transforms can be factorized into a sequence of elementary reversible matrices that map integers to integers. These elementary reversible matrices can be further factorized into lifting matrices, which allow us to generalize the lifting scheme to a more flexible framework. Our future work will focus on finding the minimal, with respect to the number of lifting factors, factorization for M-band wavelets with $L \geqslant 4$.

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