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An algebraic construction of orthonormal M-band wavelets with perfect reconstruction ☆

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Abstract

This paper presents an algebraic approach to construct M-band orthonormal wavelet bases with perfect reconstruction. We first derive the system of constraint equations of M-band filter banks, and then an algebraic solution based on matrix decomposition is developed. The structure of the solutions is presented, and practical construction procedures are given. By using this algebraic approach, some well-known K-regular M-band filter banks are constructed. The advantage of our approach is that more flexibility can be achieved, and hence we can select the best wavelet bases for a general purpose or a particular application.

Keywords: Wavelet transforms; M-band wavelets; Perfect reconstruction; Algebraic approach

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1. Introduction

It is well known that 2-band orthogonal wavelets [4,12] suffer from severe constraint conditions, such as nontrivial symmetric (linear-phase) 2-band orthogonal wavelets do not exist [3,8,9,17]. Biorthogonal wavelets, multiwavelets, and M-band wavelets are designed as alternatives for more freedom and flexibility [3,9]. M-band wavelets have attracted considerable attentions due to their richer parameter space to have a more flexible time-frequency tiling, to zoom in onto narrow band high frequency components in frequency responses, to give better energy compaction than 2-band wavelets [3,5,15]. Also, there is a close relationship between M-band wavelets and FIR perfect reconstruction filter banks [3,15,16,20,21]. The design of M-band wavelets is very challenging because of the great number of parameters and increased degrees of freedom, especially for larger M and longer length.

An M-band wavelet system consists of one scaling filter (or scaling function) and $M-1$ wavelet filters. Because most useful properties are only related to the scaling filter, a typically two-step construction procedure is used. The first step is to design the scaling filter carefully, and the second is to choose the wavelet filters from the given scaling filter [7,15,17,19]. K-regularity, linear-phase, interpolation, linear independence, local linearity, and other properties on scaling functions are investigated in [1,2,7,14,15,17,18]. For designing wavelet filters from the given scaling filter, polyphase decomposition [17], paraunitary factorization [7], and state-space characterization [15] are proposed. In [1] some shuffling operators are presented to construct wavelet filters with linear-phase and perfect reconstruction by using permutations of the scaling filter. In [19] characteristic Haar matrix is applied to construct M-band wavelets with an O(M) transform complexity, and in [11] the lifting scheme is extended to biorthogonal M-band filter banks. Cosine modulated design of the wavelets from the scaling filter are described in [3,6,10].

A disadvantage of above two-step construction is that it overlooks the greater variety in wavelet filters design, as only special cases of wavelet filters are provided. Another disadvantage is that it focuses on design filter banks with particular properties, and there is no scheme to construct general cases of filter banks. Following the idea presented in [13] for biorthogonal wavelets construction, in this paper we propose an algebraic approach to construct M-band wavelets by solving constraint equations, which can partially overcome above disadvantages of the classical two-step construction. With our approach, we can find the structures of the solutions and give practical construction procedure. Moreover, we can construct innumerable wavelet bases, among which we can select the best ones for practical applications.

Throughout this paper we use $\mathbb{Z}$ to denote the set of all integers. The symbol I denotes the identity matrix of size implied in context. Sometimes we also use I_p to denote the identity matrix of size p.

2. Problem

Suppose that $M \geqslant 3$ and $L \geqslant 2$ are two fixed positive integers. Consider the compactly supported M-band filter bank with length ML:

$$\{h_{mn}, 0 \leqslant m \leqslant M-1, 0 \leqslant n \leqslant ML-1\}. \tag{1}$$

Given a discrete signal $\{s_j\}_{j\in\mathbb{Z}}$ with finite energy, the decomposition and reconstruction formulas are

$$t_{ik} = \sum_{j\in\mathbb{Z}} h_{ij}s_{j+Mk}, \quad k \in \mathbb{Z},\ i = 0, 1, \ldots, M-1 \tag{2}$$

and

$$\tilde{s}_j = \sum_{k\in\mathbb{Z}} \sum_{i=0}^{M-1} h_{i,j-Mk}t_{ik}, \quad j \in \mathbb{Z}, \tag{3}$$

respectively. The constraint conditions for an orthonormal M-band filter bank with perfect reconstruction property are

- the low-pass and high-pass condition $\sum_{j=0}^{LM-1} h_{ij} = \sqrt{M}\delta_{i0}, i = 0, 1, \ldots, M-1$;
- the orthonormal condition $\sum_{j\in\mathbb{Z}} h_{ij}h_{r,j+Mk} = \delta_{ir}\delta_{k0}, k \in \mathbb{Z}, i, r = 0, 1, \ldots, M-1$;
- the perfect reconstruction condition $s_j = \tilde{s}_j, j \in \mathbb{Z}$.

Define

$$A_k = \begin{bmatrix} h_{0,Mk} & h_{0,Mk+1} & \cdots & h_{0,M(k+1)-1} \\ h_{1,Mk} & h_{1,Mk+1} & \cdots & h_{1,M(k+1)-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{M-1,Mk} & h_{M-1,Mk+1} & \cdots & h_{M-1,M(k+1)-1} \end{bmatrix}$$

for $k = 0, 1, \ldots, L-1$. Then the construction of an orthonormal M-band filter bank with perfect reconstruction property is reduced to the following problem.

Problem MLW. Find L real $M \times M$ matrices $A_0, A_1, \ldots, A_{L-1}$ such that

$$Se = \sqrt{M}e_1, \quad PP^{\mathrm{T}} = I, \quad QQ^{\mathrm{T}} = I, \tag{4}$$

where

$$S = \sum_{j=0}^{L-1} A_j, \quad e = (1, 1, \ldots, 1)^{\mathrm{T}}, \quad e_1 = (1, 0, \ldots, 0)^{\mathrm{T}}, \tag{5}$$

$$P = \begin{bmatrix} A_0 & A_1 & \cdots & A_{L-1} & & & & \\ & A_0 & \cdots & \cdots & A_{L-1} & & & \\ & & \cdots & \cdots & \cdots & \cdots & & \\ & & & A_0 & A_1 & \cdots & A_{L-1} \end{bmatrix}, \tag{6}$$

$$Q = \begin{bmatrix} A_0^{\mathrm{T}} & A_1^{\mathrm{T}} & \cdots & A_{L-1}^{\mathrm{T}} & & & & \\ & A_0^{\mathrm{T}} & \cdots & \cdots & A_{L-1}^{\mathrm{T}} & & & \\ & & \cdots & \cdots & \cdots & \cdots & & \\ & & & A_0^{\mathrm{T}} & A_1^{\mathrm{T}} & \cdots & A_{L-1}^{\mathrm{T}} \end{bmatrix}. \tag{7}$$

Obviously, this is a system of nonlinear matrix equations. Let $A_k = UD_kV$ with U and V both orthogonal matrices, and $D_k = \mathrm{diag}(0, I_{n_k}, 0)$, $n_0 + n_1 + \cdots + n_{L-1} = M$, $n_k \geqslant 0$. Then it is easy to verify that

$$A_kA_j^{\mathrm{T}} = A_k^{\mathrm{T}}A_j = 0, \quad j \neq k, \qquad \sum_{k=0}^{L-1} A_k^{\mathrm{T}}A_k = \sum_{k=0}^{L-1} A_kA_k^{\mathrm{T}} = I,$$

which implies that the orthonormal and perfect reconstruction conditions are naturally satisfied. Notice that in such a case $S = UV^{\mathrm{T}}$, if we take U and V satisfies that $UV^{\mathrm{T}}e = \sqrt{m}e_1$, then the low-pass and high-pass condition is satisfied. One way to select such two orthogonal matrices is as follows:

(i) Find a Householder matrix H such that $He = \sqrt{M}e_1$.
(ii) Let $V = HU$ for any given orthogonal matrix U.

Thus, we have proved the following theorem.

Theorem 2.1. *Problem MLW is always solvable, and it has infinite solutions.*

In addition, we can easily prove the following result:

Theorem 2.2. *If $\{A_0, A_1, \ldots, A_{L-1}\}$ is a solution of Problem MLW, then the matrix S defined by (5) is orthogonal, i.e., $S^{\mathrm{T}}S = SS^{\mathrm{T}} = I$.*

Remark 2.1. Theorem 2.2 shows that to solve Problem MLW is in essence to decompose an orthogonal matrix S, which satisfies the low-pass and high-pass condition, into L matrices: $S = A_0 + A_1 + \cdots + A_{L-1}$, such that the orthonormal condition and the perfect reconstruction condition are satisfied.

3. Construction

In this section we shall explore the structures of solutions to Problem MLW for $L = 2, 3, 4$. As shown below, it seems very difficult to give the general solutions to Problem MLW for $L \geqslant 5$.

3.1. The case of $L = 2$

First we consider the structures of solutions to Problem MLW for the case of $L = 2$. To this end, we first give the following basic result, which plays a fundamental role in this paper and can be easily proved by using Singular Value Decomposition Theorem.

Lemma 3.1. *If two $n \times n$ matrices A and B satisfy that $A^{\mathrm{T}}B = AB^{\mathrm{T}} = 0$, then there exist orthogonal matrices U and V such that*

$$U^{\mathrm{T}}AV = \begin{bmatrix} D_A & 0 \\ 0 & 0 \end{bmatrix}, \quad D_A = \mathrm{diag}(\sigma_1, \ldots, \sigma_{r_A}), \quad \sigma_1 \geqslant \cdots \geqslant \sigma_{r_A} > 0,$$

$$U^{\mathrm{T}}BV = \begin{bmatrix} 0 & 0 \\ 0 & D_B \end{bmatrix}, \quad D_B = \mathrm{diag}(\tau_1, \ldots, \tau_{r_B}), \quad \tau_1 \geqslant \cdots \geqslant \tau_{r_B} > 0,$$

where $r_A = \mathrm{rank}\, A$, $r_B = \mathrm{rank}\, B$, and $r_A + r_B \leqslant n$.

Notice that in the case of $L = 2$, Problem MLW can be more precisely stated as: to find two real $M \times M$ matrices A_0 and A_1 such that

$$\begin{aligned} &(A_0 + A_1)e = \sqrt{M}e_1, \quad A_0^{\mathrm{T}}A_0 = A_0A_1^{\mathrm{T}} = 0, \\ &A_0^{\mathrm{T}}A_0 + A_1^{\mathrm{T}}A_1 = A_0A_0^{\mathrm{T}} + A_1A_1^{\mathrm{T}} = I. \end{aligned}$$

The following theorem immediately follows Lemma 3.1.

Theorem 3.2. *For the case of $L = 2$, $\{A_0, A_1\}$ is a solution of Problem MLW if and only if the matrices have the following decompositions*:

$$A_0 = U\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix}V^{\mathrm{T}}, \quad A_1 = U\begin{bmatrix} 0 & 0 \\ 0 & I_{n_1} \end{bmatrix}V^{\mathrm{T}}, \tag{8}$$

where $n_0 + n_1 = M$, and U and V are orthogonal matrices with $UV^{\mathrm{T}}e = \sqrt{M}e_1$.

Remark 3.1. Theorem 3.2 provides us with a recipe for constructing the solution to Problem MLW with $L = 2$. In summary, the method is as follows:

Step 1. Select two nonnegative integers n_0 and n_1 with $n_0 + n_1 = M$, and choose an $M \times M$ orthogonal matrix U.
Step 2. Compute an $M \times M$ Householder matrix H such that $He = \sqrt{M}e_1$.
Step 3. Compute $V = HU$, and compute A_0 and A_1 as in (8).

3.2. The case of $L = 3$

For the case of $L = 3$ we have

Theorem 3.3. *For the case of* $L = 3$, $\{A_0, A_1, A_2\}$ *is a solution of Problem MLW if and only if the matrices have the decompositions* $A_k = US_kV^{\mathrm{T}}$, $k = 0, 1, 2$, *where*

$$\begin{aligned}
&S_0 = \operatorname{diag}(S, I_{n_0}, 0, 0, 0),\\
&S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -C \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 \end{bmatrix},\\
&S_2 = \operatorname{diag}(0, 0, 0, I_{n_2}, S),\\
&S = \operatorname{diag}(s_1, s_2, \ldots, s_r),\\
&C = \operatorname{diag}(c_1, c_2, \ldots, c_r),\\
&0 < s_i, c_i < 1, \quad c_i^2 + s_i^2 = 1,\\
&2r + n_0 + n_1 + n_2 = M
\end{aligned}$$

and U *and* V *are orthogonal matrices and satisfy that* $U(S_0 + S_1 + S_2)V^{\mathrm{T}}e = \sqrt{M}e_1$.

Proof. The proof of the sufficiency is trivial. Next we prove the necessity.

For the case of $L = 3$, Problem MLW can be stated as: to find three real $M \times M$ matrices A_0, A_1, A_2 such that

$$\begin{cases} (A_0 + A_1 + A_2)e = \sqrt{M}e_1, \\ A_0^{\mathrm{T}}A_2 = A_0A_2^{\mathrm{T}} = 0, \\ A_0^{\mathrm{T}}A_1 + A_1^{\mathrm{T}}A_2 = A_0A_1^{\mathrm{T}} + A_1A_2^{\mathrm{T}} = 0, \\ A_0^{\mathrm{T}}A_0 + A_1^{\mathrm{T}}A_1 + A_2^{\mathrm{T}}A_2 = A_0A_0^{\mathrm{T}} + A_1A_1^{\mathrm{T}} + A_2A_2^{\mathrm{T}} = I. \end{cases} \tag{9}$$

By Lemma 3.1 it follows $A_0^{\mathrm{T}}A_2 = A_0A_2^{\mathrm{T}} = 0$ that there exist two orthogonal matrices U_0 and V_0 such that

$$U_0^{\mathrm{T}}A_0V_0=\begin{bmatrix} D_0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad U_0^{\mathrm{T}}A_2V_0=\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & D_2\end{bmatrix}, \tag{10}$$

where D_0, D_2 are diagonal matrices with positive diagonal entries.

Let

$$U_0^{\mathrm{T}}A_1V_0=\begin{bmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}. \tag{11}$$

Substituting (10) and (11) into $A_0^{\mathrm{T}}A_1+A_1^{\mathrm{T}}A_2=A_0A_1^{\mathrm{T}}+A_1A_2^{\mathrm{T}}=0$ we have

$$\begin{aligned}&A_{11}=0,\quad A_{12}=0,\quad A_{21}=0,\quad A_{23}=0,\quad A_{32}=0,\\ &A_{33}=0,\quad D_0A_{13}+A_{31}^{\mathrm{T}}D_2=0,\quad D_0A_{31}^{\mathrm{T}}+A_{13}D_2=0.\end{aligned} \tag{12}$$

Thus, let the singular value decomposition of A_{22} be $U_1^{\mathrm{T}}A_{22}V_1=D_1$, and let $U=U_0\mathrm{diag}(I,U_1,I)$, $V=V_0\mathrm{diag}(I,V_1,I)$, then we have $U^{\mathrm{T}}A_0V=U_0^{\mathrm{T}}A_0V_0$, $U^{\mathrm{T}}A_2V=U_0^{\mathrm{T}}A_2V_0$, and

$$U^{\mathrm{T}}A_1V=\begin{bmatrix} 0 & 0 & A_{13}\\ 0 & D_1 & 0\\ A_{31} & 0 & 0\end{bmatrix}.$$

Consequently, this, together with (12) and the last two equations of (9), gives rise to

$$D_1=I,\quad A_{13}=-D_0^{-1}A_{31}^{\mathrm{T}}D_2,\quad A_{31}D_0^2=D_2^2A_{31}, \tag{13}$$

$$D_0^2+A_{31}^{\mathrm{T}}A_{31}=I,\quad D_2^2+A_{31}A_{31}^{\mathrm{T}}=I \tag{14}$$

from which we can prove that

(a) D_0 and D_2 have at least one equal diagonal entry.
(b) Suppose that $D_0=\mathrm{diag}(\sigma_1,\ldots,\sigma_\mu,\ldots,\sigma_{n_0}), D_2=\mathrm{diag}(\tau_1,\ldots,\tau_v,\ldots,\tau_{n_2})$, where $\sigma_1=\cdots=\sigma_\mu=s=\tau_1=\cdots=\tau_v$ and σ_i, $\tau_j\neq s$ if $i>\mu$, $j>v$, and let $A_{31}=\begin{bmatrix} X_{11} & X_{12}\\ X_{21} & X_{22}\end{bmatrix}$, then
  (1) when $s=1$, $X_{11}=0$, $X_{12}=0$, $X_{21}=0$;
  (2) when $0<s<1$, $\mu=v$, $X_{11}X_{11}^{\mathrm{T}}=X_{11}^{\mathrm{T}}X_{11}=c^2I$, $X_{12}=0$, and $X_{21}=0$, where $c^2=1-s^2$.

Based on the above properties of D_0, D_2 and A_{31}, if they satisfy (13) and (14), there exist two permutation matrices P and Q such that

$$
\begin{aligned}
&P^{\mathrm{T}}D_0P = \mathrm{diag}\big(s_1I_{r_1}, s_2I_{r_2}, \ldots, s_\mu I_{r_\mu}, I_{n_0}\big), \quad r_1 + r_2 + \cdots + r_\mu + n_0 = \mathrm{rank}(A_0),\\
&Q^{\mathrm{T}}D_2Q = \mathrm{diag}\big(I_{n_2}, s_1I_{r_1}, s_2I_{r_2}, \ldots, s_\mu I_{r_\mu}\big), \quad r_1 + r_2 + \cdots + r_\mu + n_2 = \mathrm{rank}(A_2),\\
&Q^{\mathrm{T}}A_{31}P = \mathrm{diag}\big(0, X_{11}, X_{22}, \ldots, X_{\mu\mu}\big), \quad X_{ii}^{\mathrm{T}}X_{ii} = (1 - s_i^2)I_{r_i},\\
&\ 0 < s_i < 1, \quad i = 1, \ldots, \mu.
\end{aligned}
$$

Thus, the necessity immediately follows. □

Remark 3.2. Based on Theorem 3.3, we can construct the solution to Problem MLW with $L = 3$ as follows:

Step 1. Select three nonnegative integers r, n_0 and n_2 with $2r + n_0 + n_2 \leqslant M$ and $2r$ real numbers s_i and c_i with $0 < s_i,\ c_i < 1$ and $s_i^2 + c_i^2 = 1$, $i = 1, 2, \ldots, r$, let $n_1 = M - 2r - n_0 - n_2$, and choose an $M \times M$ orthogonal matrix U.
Step 2. Compute an $M \times M$ Householder matrix H such that $He = \sqrt{M}e_1$.
Step 3. Construct S_0, S_1, S_2 as in Theorem 3.3 and compute $V = HUS$, where $S = S_0 + S_1 + S_2$.
Step 4. Compute $A_k = US_kV$ for $k = 0$, 1, 2.

3.3. The case of L = 4

Finally, we consider the case of $L = 4$. In this case, similar to the proof of Theorem 3.3 we can prove the following theorem.

Theorem 3.4. *For the case of* $L = 4$, $\{A_0, A_1, A_2, A_3\}$ *is a solution of Problem MLW if and only if the matrices have the decompositions* $A_k = US_kV^{\mathrm{T}}$, $k = 0$, 1, 2, 3, *where*

$$
S_0 = \mathrm{diag}(D_0, 0, 0), \quad S_1 = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & 0 \\ B_{31} & 0 & 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix},
$$

$$
S_3 = \mathrm{diag}(0, 0, D_3), \quad S_2 = \begin{bmatrix} 0 & 0 & C_{13} \\ 0 & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}, \quad C_{22} = \begin{bmatrix} 0 & 0 \\ 0 & D_2 \end{bmatrix},
$$

and moreover, U *and* V *are orthogonal matrices and satisfy that* $U(S_0 + S_1 + S_2 + S_3)V^{\mathrm{T}}e = \sqrt{M}e_1$, $D_i (i = 0, 1, 2, 3)$ *are diagonal matrices with positive entries,* $n_0 + n_3 \leqslant M$, $n_1 + n_2 \leqslant M - n_0 - n_3$, $n_i = \mathrm{size}(D_i)$, *while submatrices* B_{ij}, C_{ij}, D_i *satisfy the following system of* 18 *matrix equations*:

$$C_{13}=-D_0^{-1}B_{31}^{\mathrm{T}}D_3,\quad B_{13}=-D_0C_{31}^{\mathrm{T}}D_3^{-1},\quad C_{33}=-B_{31}C_{31}^{\mathrm{T}}D_3^{-1},$$

$$B_{11}=-D_0^{-1}B_{31}^{\mathrm{T}}C_{31},\quad C_{23}=-\left(B_{21}C_{31}^{\mathrm{T}}+B_{22}C_{32}^{\mathrm{T}}\right)D_3^{-1},$$

$$B_{12}=-D_0^{-1}\left(B_{21}^{\mathrm{T}}C_{22}+B_{31}^{\mathrm{T}}C_{32}\right),$$

$$C_{31}^{\mathrm{T}}F_BD_3^2-\left(D_0^2+B_{21}^{\mathrm{T}}B_{21}+B_{31}^{\mathrm{T}}B_{31}\right)C_{31}^{\mathrm{T}}-B_{21}^{\mathrm{T}}B_{22}C_{32}^{\mathrm{T}}=0,$$

$$D_0^2F_CB_{31}^{\mathrm{T}}-B_{31}^{\mathrm{T}}\left(C_{31}C_{31}^{\mathrm{T}}+C_{32}C_{32}^{\mathrm{T}}+D_3^2\right)-B_{21}^{\mathrm{T}}C_{22}C_{32}^{\mathrm{T}}=0,$$

$$C_{32}^{\mathrm{T}}F_BD_3^2-B_{22}^{\mathrm{T}}B_{22}C_{32}^{\mathrm{T}}+C_{22}^{\mathrm{T}}B_{21}D_0^{-2}B_{31}^{\mathrm{T}}D_3^2-B_{22}^{\mathrm{T}}B_{21}C_{31}^{\mathrm{T}}=0,$$

$$D_0^2F_CB_{21}^{\mathrm{T}}-B_{21}^{\mathrm{T}}C_{22}C_{22}^{\mathrm{T}}-B_{31}^{\mathrm{T}}C_{32}C_{22}^{\mathrm{T}}+D_0^2C_{31}^{\mathrm{T}}D_3^{-2}C_{32}B_{22}^{\mathrm{T}}=0,$$

$$C_{32}^{\mathrm{T}}F_BC_{31}+B_{22}^{\mathrm{T}}B_{21}+C_{22}^{\mathrm{T}}B_{21}D_0^{-2}B_{31}^{\mathrm{T}}C_{31}=0,$$

$$B_{31}F_CB_{21}^{\mathrm{T}}+C_{32}C_{22}^{\mathrm{T}}+B_{31}C_{31}^{\mathrm{T}}D_3^{-2}C_{32}B_{22}^{\mathrm{T}}=0,$$

$$C_{31}^{\mathrm{T}}F_BC_{31}+B_{21}^{\mathrm{T}}B_{21}+B_{31}^{\mathrm{T}}B_{31}+D_0^2=I,$$

$$B_{31}F_CB_{31}^{\mathrm{T}}+C_{31}C_{31}^{\mathrm{T}}+C_{32}C_{32}^{\mathrm{T}}+D_3^2=I,$$

$$C_{32}^{\mathrm{T}}C_{32}+B_{22}^{\mathrm{T}}B_{22}+C_{22}^{\mathrm{T}}C_{22}+\left(C_{22}^{\mathrm{T}}B_{21}+C_{32}^{\mathrm{T}}B_{31}\right)D_0^{-2}\left(B_{21}^{\mathrm{T}}C_{22}+B_{31}^{\mathrm{T}}C_{32}\right)=I,$$

$$B_{21}B_{21}^{\mathrm{T}}+B_{22}B_{22}^{\mathrm{T}}+C_{22}C_{22}^{\mathrm{T}}+\left(B_{21}C_{31}^{\mathrm{T}}+B_{22}C_{32}^{\mathrm{T}}\right)D_3^{-2}\left(C_{31}B_{21}^{\mathrm{T}}+C_{32}B_{22}^{\mathrm{T}}\right)=I,$$

$$D_3^2F_BD_3^2+C_{31}\left(D_0^2+B_{31}^{\mathrm{T}}B_{31}\right)C_{31}^{\mathrm{T}}+\left(C_{31}B_{21}^{\mathrm{T}}+C_{32}B_{22}^{\mathrm{T}}\right)\left(B_{21}C_{31}^{\mathrm{T}}+B_{22}C_{32}^{\mathrm{T}}\right)=D_3^2,$$

$$D_0^2F_CD_0^2+B_{31}^{\mathrm{T}}\left(D_3^2+C_{31}C_{31}^{\mathrm{T}}\right)B_{31}+\left(B_{21}^{\mathrm{T}}C_{22}+B_{31}^{\mathrm{T}}C_{32}\right)\left(C_{22}^{\mathrm{T}}B_{21}+C_{32}^{\mathrm{T}}B_{31}\right)=D_0^2,$$

where

$$F_B=I+B_{31}D_0^{-2}B_{31}^{\mathrm{T}},\quad F_C=I+C_{31}^{\mathrm{T}}D_3^{-2}C_{31}.$$

Although it seems not easy to find solutions to the above system, from it we can derive some explicit formulas to construct filter banks for small M. As an application of the above result, consider the case of $M = 4$, and assume that the matrices S_i in Theorem 3.4 have the following forms:

$$S_0=\left[\begin{array}{c|cc|c} \alpha & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 \end{array}\right],\quad S_3=\left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & \delta \end{array}\right],$$

$$S_1=\left[\begin{array}{c|cc|c} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14}\\ \hline \beta_{21} & \beta & 0 & 0\\ \beta_{31} & 0 & 0 & 0\\ \hline \beta_{41} & 0 & 0 & 0 \end{array}\right],\quad S_2=\left[\begin{array}{c|cc|c} 0 & 0 & 0 & \gamma_{14}\\ \hline 0 & 0 & 0 & \gamma_{24}\\ 0 & 0 & \gamma & \gamma_{34}\\ \hline \gamma_{41} & \gamma_{42} & \gamma_{43} & \gamma_{44} \end{array}\right],$$

where $n_0 = n_1 = n_2 = n_3 = 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$. Straightforward calculations lead to the construction of those matrices S_i as follows:

Step 1. Choose four numbers δ, γ, β, γ_{41} with $0 < \delta, \gamma, \beta, \gamma_{41} < 1$ and $\beta^2 \geqslant \delta^2 + \gamma_{41}^2$;

Step 2. Find the other numbers by the following formulas:

$$\alpha = \frac{\delta\gamma}{\beta}, \quad \beta_{41}^2 = \frac{\alpha^2(\beta^2 - \delta^2 - \gamma_{41}^2)}{\delta^2 + \gamma_{41}^2}, \quad \gamma_{42}^2 = \frac{(1-\beta^2)(\delta^2 + \gamma_{41}^2)}{\beta^2},$$

$$\beta_{31}^2 = \frac{(1-\gamma^2)(\alpha^2 + \beta_{41}^2)}{\gamma^2}, \quad \gamma_{43} = -\frac{\gamma\beta_{31}\beta_{41}}{\alpha^2 + \beta_{41}^2}, \quad \beta_{21} = -\frac{\beta\gamma_{41}\gamma_{42}}{\delta^2 + \gamma_{41}^2},$$

$$\gamma_{14} = -\frac{\delta\beta_{41}}{\alpha}, \quad \beta_{14} = -\frac{\alpha\gamma_{41}}{\delta}, \quad \gamma_{24} = -\frac{\beta_{21}\gamma_{41} + \beta\gamma_{42}}{\delta},$$

$$\beta_{13} = -\frac{\beta_{31}\gamma + \beta_{41}\gamma_{43}}{\alpha}, \quad \gamma_{34} = -\frac{\beta_{31}\gamma_{41}}{\delta}, \quad \beta_{12} = -\frac{\beta_{41}\gamma_{42}}{\alpha},$$

$$\gamma_{44} = -\frac{\beta_{41}\gamma_{41}}{\delta}, \quad \beta_{11} = -\frac{\beta_{41}\gamma_{41}}{\alpha}.$$

4. Examples

In order to show the practicability of our algebraic construction of M-band wavelets, some well-known K-regular M-band wavelets [15,17] are constructed by using our method.

Example 4.1. 2-regular 3-band wavelets. For $L = 2$ and $M = 3$ we can impose the regularity constraints on the solutions of Problem MLW to obtain the well-known filter bank (Table 1) [17].

For this filter bank the associated decompositions as in Theorem 3.2 are

Table 1
2-Regular 3-band filter bank

$h_{0,n}$	$h_{1,n}$	$h_{2,n}$
0.33838609728386	−0.11737701613483	0.40363686892892
0.53083618701374	0.54433105395181	−0.62853936105471
0.72328627674361	−0.01870574735313	0.46060475252131
0.23896417190576	−0.69911956479289	−0.40363686892892
0.04651408217589	−0.13608276348796	−0.07856742013185
−0.14593600755399	0.42695403781698	0.24650202866523

$$A_0 = \begin{bmatrix} 0.3384 & 0.5308 & 0.7233 \\ -0.1174 & 0.5443 & -0.0187 \\ 0.4036 & -0.6285 & 0.4606 \end{bmatrix} = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^{\mathrm{T}},$$

$$A_1 = \begin{bmatrix} 0.2390 & 0.0465 & -0.1459 \\ -0.6991 & -0.1361 & 0.4270 \\ -0.4036 & -0.0786 & 0.2465 \end{bmatrix} = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{\mathrm{T}},$$

where

$$U = \begin{bmatrix} -0.28 & -0.85 & 0.45 \\ 0.83 & -0.45 & -0.33 \\ 0.48 & 0.28 & 0.83 \end{bmatrix}, \quad V = \begin{bmatrix} -0.84 & -0.12 & 0.53 \\ -0.16 & -0.87 & -0.46 \\ 0.51 & -0.48 & 0.71 \end{bmatrix}.$$

Example 4.2. 3-regular 3-band wavelets. Using Theorem 3.3, the example filter bank of 3-regular 3-band in [17] can be given by the following decompositions:

$$A_0 = \begin{bmatrix} 0.2031 & 0.4232 & 0.7073 \\ -0.4214 & 0.7247 & 0.0395 \\ 0.0284 & -0.1610 & -0.1081 \end{bmatrix} = U \begin{bmatrix} 0.6807 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{\mathrm{T}},$$

$$A_1 = \begin{bmatrix} 0.4462 & 0.1986 & -0.1772 \\ -0.3270 & -0.2744 & 0.3241 \\ 0.4675 & -0.2401 & 0.4897 \end{bmatrix} = U \begin{bmatrix} 0 & 0 & -0.7326 \\ 0 & 0 & 0 \\ 0.7326 & 0 & 0 \end{bmatrix} V^{\mathrm{T}},$$

$$A_2 = \begin{bmatrix} -0.0720 & -0.0444 & 0.0473 \\ -0.0681 & -0.0420 & 0.0447 \\ -0.4959 & -0.3061 & 0.3255 \end{bmatrix} = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.6807 \end{bmatrix} V^{\mathrm{T}},$$

where

$$U = \begin{bmatrix} -0.6948 & 0.7049 & -0.1424 \\ 0.7192 & 0.6817 & -0.1346 \\ 0.0022 & 0.1960 & -0.9806 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.6525 & -0.1496 & 0.7429 \\ 0.3332 & 0.8238 & 0.4585 \\ -0.6806 & 0.5467 & -0.4877 \end{bmatrix}.$$

Example 4.3. 4-regular 4-band wavelets. The 4-regular 4-band wavelet matrix (Table 2) is constructed by using the scaling filter in [15] and the paraunitary factorization method in [7] (with rank-4 DCT for its characteristic Haar matrix).

Table 2
4-Regular 4-band filter bank

$h_{0,n}$	$h_{1,n}$	$h_{2,n}$	$h_{3,n}$
0.0857130200	−0.1045086525	0.2560950163	0.1839986022
0.1931394393	0.1183282069	−0.2048089157	−0.6622893130
0.3491805097	−0.1011065044	−0.2503433230	0.6880085746
0.5616494215	−0.0115563891	−0.2484277272	−0.1379502447
0.4955029828	0.6005913823	0.4477496752	0.0446493766
0.4145647737	−0.2550401616	0.0010274000	−0.0823301969
0.2190308939	−0.4264277361	−0.0621881917	−0.0923899104
−0.1145361261	−0.0827398180	0.5562313118	−0.0233349758
−0.0952930728	0.0722022649	−0.2245618041	0.0290655661
−0.1306948909	0.2684936992	−0.3300536827	0.0702950474
−0.0827496793	0.1691549718	−0.2088643503	0.0443561794
0.0719795354	−0.4437039320	0.2202951830	−0.0918374833
0.0140770701	0.0849964877	0.0207171125	0.0128845052
0.0229906779	0.1388163056	0.0338351983	0.0210429802
0.0145382757	0.0877812188	0.0213958651	0.0133066389
−0.0190928308	−0.1152813433	−0.0280987676	−0.0174753464

Applying Theorem 3.4, we can also construct this filter bank by the following decompositions:

$$A_k = US_kV^{\mathrm{T}}, \quad k = 0, 1, 2, 3,$$

where

$$S_0 = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 0.7940 & 0 & \\ 0 & 0 & 0.2891 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.0681 & -0.0651 & 0.5935 \\ 0 & 0.4959 & 0.4639 & 0.2064 \\ 0 & -0.3152 & 0.8186 & 0 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0911 \\ 0 & 0 & 0 & -0.6500 \\ 0 & -0.1716 & -0.1639 & 0.3487 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2296 \end{bmatrix},$$

$$U = \begin{bmatrix} -0.0890 & -0.8479 & -0.4983 & -0.1572 \\ 0.1717 & 0.0031 & 0.2636 & -0.9492 \\ -0.0339 & 0.5267 & -0.8172 & -0.2314 \\ -0.9805 & 0.0593 & 0.1197 & -0.1439 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.2147 & 0.0917 & -0.8907 & -0.3900 \\ 0.6595 & -0.3911 & 0.0797 & -0.6370 \\ -0.7146 & -0.4880 & 0.2984 & -0.4028 \\ 0.0917 & -0.7749 & -0.3335 & 0.5290 \end{bmatrix}.$$

5. Conclusion

In this paper we propose an algebraic approach to construct M-band orthonormal wavelet bases with perfect reconstruction. The structure of solutions to M-band filter banks is presented, and practical construction procedures are given. We also give some examples to show how to construct K-regular M-band filter banks by using our algebraic approach. Future work is how to construct longer filter banks, how to integrate useful properties such as linear-phase, and how to select the best wavelet bases for image compression.

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