FACTORING M-BAND WAVELET TRANSFORMS INTO REVERSIBLE INTEGER MAPPINGS AND LIFTING STEPS

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ABSTRACT

In this paper, a matrix factorization method is presented for reversible integer M-band wavelet transforms. Based on an algebraic construction of orthonormal M-band wavelets with perfect reconstruction, the polyphase matrix can be factorized into a finite sequence of elementary reversible matrices that map integers to integers reversibly. We show that the reversible integer mapping is essentially equivalent to the lifting scheme, thus we extend the classical lifting scheme to a more flexible framework.

1. INTRODUCTION

For signal processing and pattern recognition, M-band wavelets have attracted considerable attention due to their ability to provide much more freedom than the classical two-band wavelets, and their close connection to M-channel filter bank. However, the increased degrees of freedom make it challenging to design M-band wavelets with some useful properties.

As there are M-1 wavelet filters and only one scaling filter in an M-band wavelet system, usually two-step construction procedure is applied to reduce the design difficulties. The first step is to design the scaling filter with K-regularity [8, 14], linear-phase [1], and other properties [2, 13]. Then in the second step, wavelet filters are chosen to meet some pre-specified conditions with the given scaling filter [1, 8, 14, 18]. The disadvantage of two-step construction is that the scaling filter and the wavelet filters are designed separately, and it can not fully exploit the freedom provided by M-band wavelets. In [4, 10, 12], the lifting scheme is generalized to M-band wavelets. But due to the complexity, it is not easy to factorize a general M-band wavelet transform into lifting steps.

In [11] an algebraic construction of orthonormal M-band wavelets with perfect reconstruction is presented based on matrix decomposition. It is natural to factorize the construction matrices further into lifting steps, or into elementary reversible matrices that immediately map integers to integers, which is proposed in this paper. In Section 2 we review some conclusions of the algebraic construction method [11], reversible integer mapping [3, 7], and the lifting scheme [5, 9]. Section 3 describes the main results of our factorization, and Section 4 shows our factorization is equivalent to the lifting scheme. This paper is concluded in Section 5.

2. PRELIMINARY

2.1. M-band wavelets

Suppose the filter bank matrix of M-band wavelets with length ML is $\boldsymbol{A}=[\boldsymbol{A}_0,\boldsymbol{A}_1,\cdots,\boldsymbol{A}_{L-1}]$, where $\boldsymbol{A}_j$ are $M\times M$ matrices with $M\ge 2$ and $L\ge 2$. The first row of $\boldsymbol{A}$ is for the low-pass filter, and other M-1 rows are for high-pass filters of the wavelets. Thus, the polyphase matrix is $\boldsymbol{P}(z)=\boldsymbol{A}_0+\boldsymbol{A}_1z^{-1}+\cdots+\boldsymbol{A}_{L-1}z^{-(L-1)}$.

The constraint conditions for an orthonormal M-band filter bank with perfect reconstruct property are as follows:

$$\begin{cases}\hat{\boldsymbol{S}}\boldsymbol{e}=\sqrt{M}\boldsymbol{e}_1\\ \boldsymbol{P}\boldsymbol{P}^T=\boldsymbol{I}\\ \boldsymbol{Q}\boldsymbol{Q}^T=\boldsymbol{I}\end{cases}\qquad(1)$$

where

$$\hat{\boldsymbol{S}}=\sum_{i=0}^{L-1}\boldsymbol{A}_i,\ \boldsymbol{e}=[1,1,\cdots,1]^T,\ \boldsymbol{e}_1=[1,0,\cdots,0]^T$$

$$\boldsymbol{P}=\begin{bmatrix}\boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{L-1} & & & \\ & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{L-1} & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{L-1}\end{bmatrix}$$

$$\boldsymbol{Q}=\begin{bmatrix}\boldsymbol{A}_0^T & \boldsymbol{A}_1^T & \cdots & \boldsymbol{A}_{L-1}^T & & & \\ & \boldsymbol{A}_0^T & \boldsymbol{A}_1^T & \cdots & \boldsymbol{A}_{L-1}^T & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & \boldsymbol{A}_0^T & \boldsymbol{A}_1^T & \cdots & \boldsymbol{A}_{L-1}^T\end{bmatrix}$$

For the case of L=2 and L=3, the following results have been proved in [11]:

- *For L=2,* $\boldsymbol{A}=[\boldsymbol{A}_0,\boldsymbol{A}_1]$ *satisfy (1) if and only if they have the following decompositions:*

$$\boldsymbol{A}_0=\boldsymbol{U}\begin{bmatrix}\boldsymbol{I}_{n_0} & 0\\ 0 & 0\end{bmatrix}\boldsymbol{V}^T,\quad \boldsymbol{A}_1=\boldsymbol{U}\begin{bmatrix}0 & 0\\ 0 & \boldsymbol{I}_{n_1}\end{bmatrix}\boldsymbol{V}^T\qquad(2)$$

where $n_0+n_1=M$ *, and* $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *are orthogonal matrices with* $\boldsymbol{U}\boldsymbol{V}^T\boldsymbol{e}=\sqrt{M}\boldsymbol{e}_1$.

[*] This work was supported by NSFC project 60302005, FANEDD of China under Grant 200038, and KFAS ISEF.

- *For L=3,* $\boldsymbol{A} = [\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2]$ *satisfy (1) if and only if they have the decompositions* $\boldsymbol{A}_k = \boldsymbol{U}\boldsymbol{S}_k\boldsymbol{V}^T$, $k = 0,1,2$, *where*

$$\boldsymbol{S}_0 = diag(\boldsymbol{S}, \boldsymbol{I}_{n0}, 0,0,0),\ \boldsymbol{S}_2 = diag(0,0,0,\boldsymbol{I}_{n2}, \boldsymbol{S}),$$

$$\boldsymbol{S}_1 = \begin{bmatrix} & & & & -\boldsymbol{C} \\ & & & 0 & \\ & & \boldsymbol{I}_{n1} & & \\ & 0 & & & \\ \boldsymbol{C} & & & & \end{bmatrix}, \quad \begin{array}{l} \boldsymbol{S} = diag(s_1, s_2, \cdots, s_r), \\ \boldsymbol{C} = diag(c_1, c_2, \cdots, c_r), \\ 0 < s_i, c_i < 1, \\ c_i^2 + s_i^2 = 1, \\ 2r + n_0 + n_1 + n_2 = M, \end{array} \tag{3}$$

and $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *are orthonormal matrices and satisfy* $\boldsymbol{U}(\boldsymbol{S}_0 + \boldsymbol{S}_1 + \boldsymbol{S}_2)\boldsymbol{V}^T\boldsymbol{e} = \sqrt{M}\boldsymbol{e}_1$.

2.2. Reversible integer mapping

An *integer factor* is defined as ±1 for real numbers. A *triangular elementary reversible matrix* (TERM) is an upper or lower triangular square matrix with integer factor diagonal entries, and a *single-row elementary reversible matrix* (SERM) is a square matrix with integer factor diagonal entries and only one row of off-diagonal entries that are not all zeros. If all the diagonal entries are equal to 1, the matrix is called a unit TERM or a unit SERM.

One important property for *elementary reversible matrices* is that we can use reversible integer mappings to approximate to them. For example, let $\boldsymbol{A} = [a_{ij}]$ is an $M \times M$ upper TERM, the linear transform $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ can be approximated by the following reversible integer mapping:

$$\begin{cases} y_i = a_{ii}x_i + \left\lfloor \sum_{j=i+1}^{M} a_{ij}x_j \right\rfloor, & i = 1,2,\cdots,M-1 \\ y_M = a_{MM}x_M \end{cases}$$

where $\lfloor r \rfloor$ denotes the integer part of a real number r. Because a_{ii} is an integer factor that does not change the magnitude, the output y_i is an integer if the input x_i is an integer. Moreover, x_i can be recovered from y_i with the order $x_M, x_{M-1}, \cdots, x_1$.

The following result shows that normalized matrices with determinant ±1 can be factorized into TERMs or SERMs, which has been proved in [7]:

Lemma 1. *If an* $M \times M$ *matrix* ***A*** *satisfies that det(**A**)= ±1, then **A** has a unit TERM factorization of **A**=**PLUS$_0$** and a unit SERM factorization of **A**=**PS$_M$S$_{M-1}$…S$_1$S$_0$** , where **P** is a permutation matrix with det(**A**)=det(**P**), **L** a unit lower TERM, **U** a unit upper TERM, **S$_0$** a unit SERM with nonzero off-diagonal entries in the last row, and **S$_m$** (m= M,M-1,...,1) a unit SERM with nonzero off-diagonal entries in the m-th row.*

2.3. The lifting scheme

The lifting scheme was developed to construct second generation wavelets [16, 17], but it was found later that first generation wavelets can be also built with the lifting scheme [5]. The lifting scheme leads to fast, reversible, in-place implementation of wavelet transforms. We will show one example to illustrate the main idea.

Consider the two-band Daubechies 4 wavelet transform [5, 9]. The filter form is

$$\begin{cases} h = \frac{1}{4\sqrt{2}}[1+\sqrt{3},\ 3+\sqrt{3},\ 3-\sqrt{3},\ 1-\sqrt{3}] \\ g = \frac{1}{4\sqrt{2}}[1-\sqrt{3},\ -3+\sqrt{3},\ 3+\sqrt{3},\ -1-\sqrt{3}] \end{cases}$$

The polyphase matrix for the filter can be formulated as

$$\tilde{\boldsymbol{P}}(z) = \begin{bmatrix} H_e(z) & H_o(z) \\ G_e(z) & G_o(z) \end{bmatrix} = \frac{1}{4\sqrt{2}} \begin{bmatrix} (1+\sqrt{3})+(3-\sqrt{3})z^{-1} & (3+\sqrt{3})+(1-\sqrt{3})z^{-1} \\ (1-\sqrt{3})+(3+\sqrt{3})z^{-1} & (-3+\sqrt{3})+(-1-\sqrt{3})z^{-1} \end{bmatrix}.$$

The determinant of the polyphase matrix is $-z^{-1}$. Usually the normalized polyphase matrix with determinant 1 is used, which can be given by

$$\boldsymbol{P}(z) = \frac{1}{4\sqrt{2}} \begin{bmatrix} (1+\sqrt{3})+(3-\sqrt{3})z^{-1} & (3+\sqrt{3})+(1-\sqrt{3})z^{-1} \\ -(1-\sqrt{3})z-(3+\sqrt{3}) & -(-3+\sqrt{3})z-(-1-\sqrt{3}) \end{bmatrix}.$$

Then, a lifting factorization can be given by

$$\boldsymbol{P}(z) = \begin{bmatrix} \frac{\sqrt{3}-1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{3}+1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & -z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{4} - \frac{\sqrt{3}-2}{4}z & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{bmatrix}.$$

Let the z-transform of an input signal $s[n]$ be $S(z)$, and its even and odd components are $S_e(z)$ and $S_o(z)$. Then, the z-transform representation of wavelet transform is given by

$$\begin{bmatrix} \tilde{S}(z) \\ \tilde{D}(z) \end{bmatrix} = \boldsymbol{P}(z) \begin{bmatrix} S_e(z) \\ S_o(z) \end{bmatrix}.$$

Let

$$\begin{bmatrix} S^{(1)}(z) \\ D^{(1)}(z) \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S_e(z) \\ S_o(z) \end{bmatrix}$$

where

$$S_e(z) = \sum_n s^{(0)}[n]z^{-n} = \sum_n s[2n]z^{-n}$$

$$S_o(z) = \sum_n d^{(0)}[n]z^{-n} = \sum_n s[2n+1]z^{-n}$$

$$S^{(1)}(z) = \sum_n s^{(1)}[n]z^{-n},\ D^{(1)}(z) = \sum_n d^{(1)}[n]z^{-n}.$$

By the uniqueness of the z-transform representation, we have

$$\begin{cases} s^{(1)}[n] = s^{(0)}[n] + \sqrt{3}d^{(0)}[n] \\ d^{(1)}[n] = d^{(0)}[n] \end{cases}$$

Sequentially, we obtain the following lifting steps:

$$\begin{cases} s^{(2)}[n] = s^{(1)}[n] \\ d^{(2)}[n] = -\frac{\sqrt{3}}{4}s^{(1)}[n] - \frac{\sqrt{3}-2}{4}s^{(1)}[n+1] + d^{(1)}[n] \end{cases}$$

$$\begin{cases} s^{(3)}[n] = s^{(2)}[n] - d^{(2)}[n-1] \\ d^{(3)}[n] = d^{(2)}[n] \end{cases}, \quad \begin{cases} s^{(4)}[n] = \frac{\sqrt{3}-1}{\sqrt{2}}s^{(3)}[n] \\ d^{(4)}[n] = \frac{\sqrt{3}+1}{\sqrt{2}}d^{(3)}[n] \end{cases}$$

From this example, we can see that the filter form, the matrix factorization, and the lifting steps can be converted from one representation into another [9]. In addition, the z^m term in the lifting factorization corresponds to $s^{(i)}[n+m]$ or $d^{(i)}[n+m]$ in the lifting steps.

3. FACTORIZATIONS

In this section, we give the TERM or SERM factorization of the polyphase matrix $P(z)$ of an orthonormal M-band filter bank $A=[A_0, A_1, \cdots, A_{L-1}]$ with perfect reconstruction for the cases of L=2 and L=3.

3.1. The case of L=2

For the case of L=2, by (2), the polyphase matrix has the following form:

$$P(z) = A_0 + A_1 z^{-1} = U \begin{bmatrix} I_{n0} & 0 \\ 0 & I_{M-n0} z^{-1} \end{bmatrix} V^T .$$

Because U and V are both orthonormal matrices, det(U)=±1 and det(V)=±1. By Lemma 1, U and V have TERM factorization of form $PLUS_0$ and SERM factorization of form $PS_M S_{M-1} \ldots S_1 S_0$. The intermediate matrix

$$\begin{bmatrix} I_{n0} & 0 \\ 0 & I_{M-n0} z^{-1} \end{bmatrix}$$

is equivalent to identity matrix, except for a translation of the input signal corresponding to the lower-right part. Thus, reversible integer mapping can be implemented for M-band wavelets of the case L=2.

3.2. The case of L=3

For the case of L=3, by (3), the polyphase matrix has the following form:

$$P(z) = A_0 + A_1 z^{-1} + A_2 z^{-2} = U \begin{bmatrix} S & & -Cz^{-1} \\ & B & \\ Cz^{-1} & & Sz^{-2} \end{bmatrix} V^T ,$$

where

$$B = \begin{bmatrix} I_{n0} & & \\ & I_{n1} z^{-1} & \\ & & I_{n2} z^{-2} \end{bmatrix} .$$

Notice that

$$\begin{bmatrix} S & & -Cz^{-1} \\ & B & \\ Cz^{-1} & & Sz^{-2} \end{bmatrix} = \begin{bmatrix} S & & -Cz^{-1} \\ & I & \\ Cz^{-1} & & Sz^{-2} \end{bmatrix} \begin{bmatrix} I & & \\ & B & \\ & & I \end{bmatrix}$$

and the transform with B can be implemented for reversible integer mapping directly, we only need to consider how to factorize the matrix

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix} .$$

Noting that C, S, and C+S are all nonsingular, $C^2 + S^2 = I$, and $SC = CS$, and using the following useful equalities:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & A_{12} \\ A_{21} + A_{22} A_{12}^{-1}(I - A_{11}) & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{12}^{-1}(I - A_{11}) & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ A_{21} A_{11}^{-1} & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & I \end{bmatrix} ,$$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{21} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix} ,$$

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix} ,$$

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & ACB^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} ,$$

$$\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/a-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/a \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1/a & 1 \end{bmatrix} \begin{bmatrix} 1 & a-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/a-1 \\ 0 & 1 \end{bmatrix} ,$$

we can get many different reversible integer mapping factorizations. For the limitation of the paper length, we here just present four of them as below.

1. The factorization with 3 TERMs:

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix} = \begin{bmatrix} I & -Cz^{-1} \\ (I-S)C^{-1}z^{-1} & Sz^{-2} \end{bmatrix} \begin{bmatrix} I & 0 \\ C^{-1}(I-S)z & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ (I-S)C^{-1}z^{-1} & Iz^{-2} \end{bmatrix} \begin{bmatrix} I & -Cz^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C^{-1}(I-S)z & I \end{bmatrix} .$$

2. The factorization with 4 TERMs:

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix} = \begin{bmatrix} -Cz^{-1} & S \\ Sz^{-2} & Cz^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & S \\ (S + CS^{-1}C)z^{-2} + CS^{-1}z^{-1} & Cz^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -S^{-1}(I + Cz^{-1}) & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ S^{-1}z^{-2} + CS^{-1}z^{-1} & -Iz^{-2} \end{bmatrix} \begin{bmatrix} I & S \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -S^{-1}(I + Cz^{-1}) & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} .$$

3. The factorization with 4 TERMs:

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & Iz^{-1} \end{bmatrix} \begin{bmatrix} S-C & -(C+S)z^{-1} \\ C & Sz^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I & I \\ 0 & Iz^{-1} \end{bmatrix} \begin{bmatrix} I & -(C+S) \\ X & S \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & Iz^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} I & I \\ 0 & Iz^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} I & -(C+S) \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ Y & Iz^{-1} \end{bmatrix} ,$$

where

$$X = C - S(C+S)^{-1}(I - S + C), \quad Y = (C+S)^{-1}(I - S + C).$$

4. The factorization with 7 TERMs:

$$\begin{bmatrix} S & -Cz^{-1} \\ Cz^{-1} & Sz^{-2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ CS^{-1}z^{-1} & S^{-1}z^{-2} \end{bmatrix} \begin{bmatrix} S & -Cz^{-1} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ CS^{-1}z^{-1} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S^{-1}z^{-2} \end{bmatrix} \begin{bmatrix} I & -Cz^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ CS^{-1}z^{-1} & I \end{bmatrix} \begin{bmatrix} I & -CSz \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S^{-1}z^{-2} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ CS^{-1}z^{-1} & I \end{bmatrix} \begin{bmatrix} I & -CSz \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Iz^{-2} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S^{-1} \end{bmatrix}$$

where

$$\begin{bmatrix} S & 0 \\ 0 & S^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ S^{-1}-I & I \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ S-I & I \end{bmatrix} \begin{bmatrix} I & -S^{-1} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ -S^{-1} & I \end{bmatrix} \begin{bmatrix} I & S-I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} I & S^{-1}-I \\ 0 & I \end{bmatrix}$$

Thus, reversible integer mapping can be implemented for M-band wavelets of the case L=3.

4. EQUIVALENCE TO THE LIFTING SCHEME

It has been proved in [5] and [12] that any biorthogonal two-band or M-band wavelet transform can be obtained using lifting. This corresponds to a factorization of the polyphase matrix into a sequence of lifting matrices and one diagonal scaling matrix. A lifting matrix is a unit upper or lower triangular square matrix with nonzero off-diagonal entries only in the first column or the first row.

We show that the reversible integer mapping is equivalent to the lifting scheme. Obviously, a lifting matrix is a TERM or a SERM. On the other hand, a TERM can be converted into a sequence of SERMs [7], a SERM can be converted into a SERM corresponding to the first row by one row exchange and one column exchange only, and a SERM or a permutation matrix can also be factorized into lifting matrices. Then essence can be conveyed by the following simple examples. Let

$$\boldsymbol{S}=\begin{bmatrix}1&&&&\\&1&&&\\a&b&1&c&d\\&&&1&\\&&&&1\end{bmatrix},\qquad \boldsymbol{P}=\begin{bmatrix}1&&&\\&1&&\\&&&1\\&&1&\end{bmatrix}.$$

Straightforward calculations verify that

$$\boldsymbol{S}=\boldsymbol{P}_{13}\boldsymbol{S}_1\boldsymbol{P}_{13},\quad \boldsymbol{S}=\boldsymbol{L}_1\boldsymbol{U}_1\boldsymbol{L}_2\boldsymbol{U}_2\boldsymbol{L}_3,\quad \boldsymbol{P}=\boldsymbol{D}(\tilde{\boldsymbol{L}}_1\tilde{\boldsymbol{U}}_1)^2,$$

where

$$\boldsymbol{P}_{13}=\begin{bmatrix}&&1&&\\&1&&&\\1&&0&&\\&&&1&\\&&&&1\end{bmatrix},\boldsymbol{S}_1=\begin{bmatrix}1&b&a&c&d\\&1&&&\\&&1&&\\&&&1&\\&&&&1\end{bmatrix},\boldsymbol{L}_1=\begin{bmatrix}1&&&&\\&1&&&\\1&&1&&\\&&&1&\\&&&&1\end{bmatrix},$$

$$\boldsymbol{U}_1=\begin{bmatrix}1&b&&c&d\\&1&&&\\&&1&&\\&&&1&\\&&&&1\end{bmatrix},\boldsymbol{L}_2=\begin{bmatrix}1&&&&\\&1&&&\\-1&&1&&\\&&&1&\\&&&&1\end{bmatrix},\boldsymbol{U}_2=\begin{bmatrix}1&-b&&-c&-d\\&1&&&\\&&1&&\\&&&1&\\&&&&1\end{bmatrix},$$

$$\boldsymbol{L}_3=\begin{bmatrix}1&&&&\\&1&&&\\a&&1&&\\&&&1&\\&&&&1\end{bmatrix},\boldsymbol{D}=\begin{bmatrix}-1&&&\\&1&&\\&&1&\\&&&1\end{bmatrix},\tilde{\boldsymbol{L}}_1=\begin{bmatrix}1&&&\\&1&&\\1&&1&\\-1&&&1\end{bmatrix},\tilde{\boldsymbol{U}}_1=\begin{bmatrix}1&&-1&1\\&1&&\\&&1&\\&&&1\end{bmatrix}.$$

5. CONCLUSION

In this paper, we first review an algebraic construction of M-band wavelets, reversible integer mapping, and the lifting scheme. Based on the algebraic construction, the polyphase matrix can be factorized into a sequence of elementary reversible matrices that map integers to integers. These elementary reversible matrices can be further factorized into lifting matrices, which establish the equivalence to the lifting scheme, and allow us to generalize the lifting scheme to a more flexible framework. To find the general and optimal factorization for generic M-band wavelets is our future work.

6. REFERENCES

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