AN ALGEBRAIC APPROACH TO M-BAND WAVELETS CONSTRUCTION

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Abstract

This paper presents an algebraic approach to construct *M*-band orthogonal wavelet bases. A system of constraint equations is obtained for *M*-band orthonormal filters, and then a solution based on SVD (Singular Value Decomposition) is developed to enable us to produce innumerable wavelet bases of given length. Also the property of 2 vanishing moments is integrated into our wavelet construction process, which provides another way to compute 2-regular *M*-band filter banks.

Key Words
Wavelet transforms, *M*-band wavelets, algebraic approach, vanishing moments

1. Introduction

It is well known that 2-band orthogonal wavelets suffer from severe constraint conditions, such as nontrivial symmetric (linear-phase) orthogonal wavelets do not exist. Biorthogonal wavelets, multiwavelets, and M-band wavelets are designed as alternatives for more freedom and capacities. This paper devotes to M-band wavelets construction.

The design of M-band linear phase filter bank has been investigated in [1]. Similar work was published in [2] but based on time-domain constraints and only for $M=2^i$. In [3] and [4], filters of *K* vanishing moments (or stated as *K*-regular) and finite length are constructed. Scaling filters are computed based on unitariness and autocorrelation, or maximal flatness condition. Then matrix extension by polyphase decomposition, or state-space characterization is used to construct the (*M*-1) wavelet filters, though not uniquely determined by the scaling filter (unlike the 2-band case). Biorthogonal *M*-band filters are designed using lifting scheme in [5].

Following the idea presented in [6] for 2-band biorthogonal wavelets, in this paper an algebraic approach is proposed to construct *M*-band wavelets by solving the constraint equations. In contrast with Fourier-based methods focusing on a few deliberate filters, our algebraic approach gives the explicit formula with some free parameters to produce innumerable wavelet bases, among which we can find appropriate ones for specific applications.

The rest of this paper is organized as follows. Section 2 is a brief review of algebraic approach on 2-band biorthogonal wavelets construction. Section 3 introduces some background knowledge of *M*-band wavelets. Our algebraic method is presented in Section 4 to construct *M*-band orthonormal wavelet bases. In Section 5, conditions of two vanishing moments are derived and integrated into our construction method, with the direct results being the well-known Daubechies scaling filters. The conclusions are summarized in Section 6.

2. Review of the algebraic approach for 2-band biorthogonal wavelets

Let *Z* denote the set of all integers and $\varphi(x)$, $\tilde{\varphi}(x)$ be two scaling functions satisfying the double scaling equations

$$\begin{cases}\varphi(x)=\sqrt{2}\sum_{n\in Z}h_n\varphi(2x-n)\\ \tilde{\varphi}(x)=\sqrt{2}\sum_{n\in Z}\tilde{h}_n\tilde{\varphi}(2x-n)\end{cases}$$

The corresponding wavelet functions are

$$\begin{cases}\psi(x)=\sqrt{2}\sum_{n\in Z}g_n\varphi(2x-n)\\ \tilde{\psi}(x)=\sqrt{2}\sum_{n\in Z}\tilde{g}_n\tilde{\varphi}(2x-n)\end{cases}$$

where $g_n=(-1)^n\tilde{h}_{1-n},\ \tilde{g}_n=(-1)^n h_{1-n},\ n\in Z$. Usually, we take $h_n,\ \tilde{h}_n$ as low-pass and $g_n,\ \tilde{g}_n$ as high-pass coefficients if

$$\sum_{n\in Z}h_n=\sum_{n\in Z}\tilde{h}_n=\sqrt{2},\ \sum_{n\in Z}g_n=\sum_{n\in Z}\tilde{g}_n=0$$

In addition, the linear phase property requires $h_{-n}=h_n,\ \tilde{h}_{-n}=\tilde{h}_n,\ n\in Z$, and the biorthogonal conditions imposed on $\varphi(x)$, $\tilde{\varphi}(x)$, $\psi(x)$ and $\tilde{\psi}(x)$ are equivalent to

$$\begin{cases}\sum_{n\in Z}h_n\tilde{h}_{n-2j}=\delta_j, & j\in Z\\ \sum_{n\in Z}g_n\tilde{g}_{n-2j}=\delta_j, & j\in Z\\ \sum_{n\in Z}h_n\tilde{g}_{n-2j}=0, & j\in Z\\ \sum_{n\in Z}g_n\tilde{h}_{n-2j}=0, & j\in Z\end{cases}$$

where $\delta_j=1$ when $j=0$ and $\delta_j=0$ otherwise.

Given a discrete signal $\{a_n\}_{n\in Z}$ with finite energy, the discrete biorthogonal wavelet transform

$$\begin{cases}c_k=\sum_{n\in Z}h_{n-2k}a_n, & k\in Z\\ d_k=\sum_{n\in Z}g_{n-2k}a_n, & k\in Z\end{cases}$$

decomposes $\{a_n\}_{n\in Z}$ into $\{c_k\}_{k\in Z}$ and $\{d_k\}_{k\in Z}$, and the inverse transform $a_n = \sum_{k\in Z} \tilde{h}_{n-2k} c_k + \sum_{k\in Z} \tilde{g}_{n-2k} d_k,\ n\in Z$ reconstructs $\{a_n\}_{n\in Z}$. Therefore, the perfect reconstruction (PR) condition is

$$\sum_{k\in Z} \tilde{h}_{n-2k} h_{m-2k} + \sum_{k\in Z} \tilde{g}_{n-2k} g_{m-2k} = \delta_{m,n}, \qquad m,n\in Z$$

which is proved to be equivalent to

$$\sum_{n\in Z} h_n \tilde{h}_{n-2j} = \delta_j, \qquad j\in Z$$

Consequently, a system of the constraint equations of biorthogonal wavelets can be derived as

$$\begin{cases} \sum_{n\in Z} h_n \tilde{h}_{n-2j} = \delta_j, & j\in Z \\ \sum_{n\in Z} h_{2n} = \sum_{n\in Z} h_{2n+1} = \frac{1}{\sqrt{2}} \\ \sum_{n\in Z} \tilde{h}_{2n} = \sum_{n\in Z} \tilde{h}_{2n+1} = \frac{1}{\sqrt{2}} \\ h_{-n} = h_n, \tilde{h}_{-n} = \tilde{h}_n, & n\in Z \end{cases}$$

Considering the 5-7 tap case as an example and let $p_n = \sqrt{2} h_n,\ q_n = \sqrt{2}\tilde{h}_n$, we have the constraint equation system as:

$$\begin{cases} p_0 q_0 + 2p_1 q_1 + 2p_2 q_2 = 2 \\ p_1 q_3 + p_0 q_2 + p_1 q_1 + p_2 q_0 = 0 \\ p_1 q_3 + p_2 q_2 = 0 \\ p_0 + 2p_2 = 1 \\ 2p_1 = 1 \\ q_0 + 2q_2 = 1 \\ 2q_1 + 2q_3 = 1 \\ p_{-n} = p_n,\ n = 1,2; \quad q_{-n} = q_n,\ n = 1,2,3 \end{cases}$$

If we take p_2 as a free parameter, then the solution is

$$\begin{cases} q_{\pm 2} = \frac{4p_2 + 1}{4(4p_2 - 1)},\ q_{\pm 3} = -2p_2 q_2,\ q_0 = 1 - 2q_2,\ q_{\pm 1} = \frac{1}{2} - q_3 \\ p_0 = 1 - 2p_2,\ p_{\pm 1} = \frac{1}{2},\ p_{-2} = p_2 \end{cases}$$

For $p_2 = -0.1$, we get the 5-7 tap Daubechies biorthogonal filters. In a similar manner, we can get the 9-7 tap Daubechies biorthogonal filters.

3. Background material on *M*-band wavelets

Let $M \ge 2$ be a fixed positive integer. A family of closed subspaces $V_j (j\in Z)$ in square integrable function space L^2 is called a *multiresolution* of L^2 if the following conditions hold:

(i) $V_j \subset V_{j-1}$, and $f\in V_j$ if and only if $f(M\bullet)\in V_{j-1}$ for all $j\in Z$;

(ii) $\bigcup_{j\in Z} V_j$ is dense in L^2 and $\bigcap_{j\in Z} V_j = \{0\}$;

(iii) $\{\phi(\bullet - n); n\in Z\}$ is an orthonormal basis of V_0 for some $\phi\in V_0$.

The function ϕ in (iii), called *M-band scaling function*, satisfies the refinement equation

$$\phi(x) = \sum_{n\in Z} c_n \phi(Mx - n)$$

where $\{c_n\}_{n\in Z}$ satisfies $\sum_{n\in Z} c_n = \sqrt{M}$. It is easy to see that

$$c_n = <\phi(x), \phi_{-1,n}(x)> = \sqrt{M}\int_R \phi(x)\phi(Mx-n)dx$$

$$\sum_{n\in Z} c_n c_{n-Mk} = \delta_{0,k}, \qquad k\in Z$$

Apparently, the orthonormal basis of V_j is

$$\{\phi_{j,n}(x) = M^{-j/2}\phi(M^{-j}x - n)\}_{n\in Z}$$

Let W_j be the orthogonal complement of V_j in V_{j-1}:

$$V_{j-1} = V_j \oplus W_j$$

It can be proved that there exist $\psi_1,\cdots,\psi_{M-1}$ such that

$$\{\psi^{(r)}{}_{j,n}(x) = M^{-j/2}\psi_r(M^{-j}x - n)\}_{n\in Z,\ 1\le r\le M-1}$$

becomes the orthogonal basis of W_j. Let

$$d_{r,n} = <\psi_r(x), \phi_{-1,n}(x)> = \sqrt{M}\int_R \psi_r(x)\phi(Mx-n)dx$$

then

$$\psi_r(x) = \sqrt{M}\sum_{n\in Z} d_{r,n}\phi(Mx-n)$$

Let $h_{0,n} = c_n, h_{r,n} = d_{r,n}, r = 1,\cdots,M-1$, and take $h_{0,n}$ as low-pass and $h_{r,n}$ as high-pass coefficients, we have:

$$\sum_{n\in Z} h_{r,n} = \sqrt{M}\delta_{r,0} \qquad 0\le r\le M-1 \tag{1}$$

Moreover, the orthonormal property requires that

$$\sum_{s=0}^{M-1}\left|\hat{h}_0(\xi + \frac{2s\pi}{M})\right|^2 = 1, \qquad \forall \xi\in R$$

$$\sum_{n\in Z} h_{r,n} h_{r',n+Mk} = \delta_{r,r'}\delta_{k,0} \qquad k\in Z,\ 0\le r,r'\le M-1 \tag{2}$$

Given a discrete signal $\{s_n\}_{n\in Z}$ with finite energy, the decomposition and reconstruction formulas are

$$\begin{cases} b_{r,k} = \sum_{j\in Z} h_{r,j} s_{j+Mk} & k\in Z,\ 0\le r\le M-1 \\ \tilde{s}_n = \sum_{k\in Z}\sum_{r=0}^{M-1} h_{r,n-Mk} b_{r,k} & n\in Z \end{cases} \tag{3}$$

Define $H(z) = \frac{1}{M}\sum_{k\in Z} c_k z^k$. For an integer $K\ge 1$, we say $H(z)$ has K vanishing moments, or $H(z)$ is K-regular, if there exists a Laurent polynomial $Q(z)$ such that

$$H(z) = \left(\frac{1 - z^M}{M(1-z)}\right)^K Q(z)$$

For a fixed positive integer k, $0\le k < K$, the definition of vanishing moments K is equivalent to any of following statements:

(i) $\hat{\phi}^{(k)}(2n\pi) = 0,\ n\in Z, n\ne 0$

(ii) $\hat{h}_0^{(k)}(\frac{2s\pi}{M}) = 0,\ \forall s = 1,\dots,M-1$

(iii) $\sum_{n\in Z} n^k h_{0,n}(e^{-i\frac{2s\pi}{M}})^n = 0,\ \forall s = 1,\dots,M-1$

With regard to (2) and $H(0) = 1$, we know that $H(z)$ has at least 1 vanishing moment.

4. Construction of *M*-band wavelets with compactly supported length *LM*

Let $L \geq 2$ be a fixed positive integer. For simplicity, we consider $M = 4$ and $L = 2$ firstly, while general cases treated later. More precisely, we consider $\{h_{r,n}\}_{0\leq r\leq 3, 0\leq n\leq 7}$.

4.1 case 1: *M*=4 and *L*=2

In order to establish the system of constraint equations on $\{h_{r,n}\}_{0\leq r\leq 3, 0\leq n\leq 7}$, perfect reconstruction conditions are needed, as stated in Theorem 1.

Theorem 1. Perfect reconstruction conditions of $\{h_{r,n}\}_{0\leq r\leq 3, 0\leq n\leq 7}$ for tranform (3) are

$$\begin{cases} \sum_{r=0}^{3} h_{r,i} h_{r,j+4} = 0 \qquad 0 \leq i,j \leq 3 \\ \sum_{r=0}^{3} (h_{r,i} h_{r,j} + h_{r,i+4} h_{r,j+4}) = \delta_{i,j} \qquad 0 \leq i,j \leq 3 \end{cases} \tag{4}$$

Proof: Let $m = [n/4]$, for $i = n - 4m, (i = 0,1,2,3)$, we have $n = 4m + i$, and

$$\tilde{s}_n = \sum_{k\in Z} \sum_{r=0}^{3} h_{r,n-4k} b_{r,k} = \sum_{r=0}^{3} \sum_{k=m-1}^{m} h_{r,n-4k} b_{r,k} = \sum_{r=0}^{3} (h_{r,i} b_{r,m} + h_{r,i+4} b_{r,m-1})$$

Let $g_{i,j} = \sum_{r=0}^{3} h_{r,i} h_{r,j}$, then

$$\begin{aligned} \tilde{s}_{4m+i} &= \sum_{r=0}^{3} \sum_{j=0}^{7} (h_{r,i} h_{r,j} s_{4m+j} + h_{r,i+4} h_{r,j} s_{4m+j-4}) \\ &= \sum_{j=0}^{7} (g_{i,j} s_{4m+j} + g_{i+4,j} s_{4m+j-4}) \\ &= \sum_{j=0}^{3} g_{i+4,j} s_{4m+j-4} + \sum_{j=0}^{3} (g_{i,j} + g_{i+4,j+4}) s_{4m+j} + \sum_{j=4}^{7} g_{i,j} s_{4m+j} \end{aligned}$$

To achieve perfect reconstruction, following equations should be satisfied:

$$\begin{cases} g_{i+4,j} = 0 \qquad 0 \leq i,j \leq 3 \\ g_{i,j} + g_{i+4,j+4} = \delta_{i,j} \qquad 0 \leq i,j \leq 3 \end{cases}$$

which are equivalent to (4). □

Theorem 2. The constraint equation set for $\{h_{r,n}\}_{0\leq r\leq 3, 0\leq n\leq 7}$ is

$$[1,1,1,1](A + B) = [2,0,0,0] \tag{5}$$

$$\begin{cases} A'B = AB' = 0 \\ A'A + B'B = AA' + BB' = I \end{cases} \tag{6}$$

where A and B are two 4×4 matrices, $(A)_{i,j} = h_{j,i}$, $(B)_{i,j} = h_{j,i+4}$. That is

$$A = \begin{bmatrix} h_{00} & h_{10} & h_{20} & h_{30} \\ h_{01} & h_{11} & h_{21} & h_{31} \\ h_{02} & h_{12} & h_{22} & h_{32} \\ h_{03} & h_{13} & h_{23} & h_{33} \end{bmatrix} \qquad B = \begin{bmatrix} h_{04} & h_{14} & h_{24} & h_{34} \\ h_{05} & h_{15} & h_{25} & h_{35} \\ h_{06} & h_{16} & h_{26} & h_{36} \\ h_{07} & h_{17} & h_{27} & h_{37} \end{bmatrix} \tag{7}$$

It can be easily proved with equation (1), (2) and (4).

The construction of 4-band filter bank $\{h_{r,n}\}_{0\leq r\leq 3, 0\leq n\leq 7}$ is reduced to solve the constraint equation set (5) and (6). Based on Singular-Value-Decomposition (SVD), we have

Theorem 3. Suppose a 4×4 diagonal matrix $D_0 = diag(d_0, d_1, d_2, d_3)$, where $d_i = 0 \text{ or } 1$, a 4×4 orthonormal (i.e. real unitary) matrix $U = [\alpha_0, \alpha_1, \alpha_2, \alpha_3]$, where α_i is the *i*-th column vector, and another 4×4 orthonormal matrix $V = [\beta_0, \beta_1, \beta_2, \beta_3]$, where

$$\beta_{i0} = \frac{1}{2}\sum_{j=0}^{3} \alpha_{ij}, \quad 0 \leq i \leq 3 \tag{8}$$

and β_{i0} is the first element of β_i, and let $D_1 = I - D_0$, then $A = UD_0V'$ and $B = UD_1V'$ are the solution of equations (5) and (6).

Proof: (i) Any $n\times n$ matrix *A* can be written in SVD form $A = UDV'$, where *U* and *V* are both $n\times n$ orthonormal matrix, $UU' = I$, $VV' = I$, U' is transpose of *U*, and *D* is a diagonal matrix with non-negative diagonal entries. Let $U = [\alpha_0, \cdots, \alpha_{n-1}]$, $V = [\beta_0, \cdots, \beta_{n-1}]$ and $D = diag(d_0, \cdots, d_{n-1})$, then we have

$$A = UDV' = \sum_{i=0}^{n-1} d_i \alpha_i \beta_i' \tag{9}$$

where α_i and β_i are *n*-dimensional column vectors, and $d_i \geq 0$. Therefore, it is easy to rearrange d_i freely, usually in descending order, if we exchange the corresponding columns in *U* and *V* simultaneously.

(ii) Let $A = UD_0V'$ and $B = UD_1V'$. Clearly, if D_0 and D_1 satisfy

$$\begin{cases} D_0 D_1 = 0 \\ D_0^2 + D_1^2 = I \end{cases} \tag{10}$$

then A and B are the solution of (6). Let $D_0 = diag(d_0, d_1, d_2, d_3)$. From $D_0^2 + D_1^2 = I$, we get $D_1 = diag(\sqrt{1-d_0^2}, \sqrt{1-d_1^2}, \sqrt{1-d_2^2}, \sqrt{1-d_3^2})$.

With regard to $D_0 D_1 = 0$, we have $d_i\sqrt{1-d_i^2} = 0, \ 0 \leq i \leq 3$. Therefore, $d_i = 0 \text{ or } 1$ for $0 \leq i \leq 3$, and $D_1 = I - D_0$.

(iii) With above results, condition (5) can be formulated as $[1,1,1,1]U(D_0 + D_1)V' = [1,1,1,1]UV' = [2,0,0,0]$, and we obtain $[1,1,1,1]U = [2,0,0,0]V$. Then, we have $\beta_{i0} = \frac{1}{2}\sum_{j=0}^{3}\alpha_{ij}, \ 0 \leq i \leq 3$, which shows that the first row of *V* is determined by *U*. Note that the first row of *V* has been normalized to 1 since

$$\begin{aligned} \sum_{i=0}^{3} \beta_{i0}^2 &= \frac{1}{4}\sum_{i=0}^{3}(\sum_{n=0}^{3}\alpha_{in})^2 \\ &= \frac{1}{4}\sum_{i=0}^{3}[1 + 2(\alpha_{i0}\alpha_{i1} + \alpha_{i0}\alpha_{i2} + \alpha_{i0}\alpha_{i3} + \alpha_{i1}\alpha_{i2} + \alpha_{i1}\alpha_{i3} + \alpha_{i2}\alpha_{i3})] \\ &= 1 + \frac{1}{2}\sum_{i=0}^{3}(\alpha_{i0}\alpha_{i1} + \alpha_{i0}\alpha_{i2} + \alpha_{i0}\alpha_{i3} + \alpha_{i1}\alpha_{i2} + \alpha_{i1}\alpha_{i3} + \alpha_{i2}\alpha_{i3}) \\ &= 1 + \frac{1}{2}(0 + 0 + 0 + 0 + 0 + 0) = 1 \end{aligned}$$

(iv) The method to design the filter banks is constructive. Firstly set $D_0 = diag(d_0, d_1, d_2, d_3)$, $d_i = 0 \text{ or } 1$ for $0 \leq i \leq 3$ and $D_1 = I - D_0$. Secondly, choose one 4×4 orthonormal matrix $U=[\alpha_0,\alpha_1,\alpha_2,\alpha_3]$. Finally, set the first row of V to meet (8), with other three rows extended by *Schmidt orthonormalization process*, we have $A=UD_0V'$ and $B=UD_1V'$ to be the solution of the equations (5) and (6). □

One question is: does theorem 3 give all the solutions for (6)? The answer is yes, as *Theorem 4* followed.

Theorem 4. For 4×4 matrices A, B and their SVD $A=U_0D_0V_0'$ and $B=U_1D_1V_1'$, if A and B are the solution of the equation system (6), then $D_1=I-D_0$, orthonormal matrices U_0 and U_1 satisfy $Q=U_0'U_1=(q_{ij})$ where $q_{ij}=0$, $0\le i\le s-1$, $s\le j\le 3$, $s=r(A)$ denotes the rank of A, and orthonormal matrices V_0 and V_1 satisfy $P=V_0'V_1=(p_{ij})$ where $p_{ij}=0$, $0\le i\le s-1$, $s\le j\le 3$.

Proof: (i) With regard to $A'B=O$, we get $r(A)+r(B)=r(A')+r(B)\le 4$. From $A'A+B'B=I$, we get $r(A)+r(B)=r(A'A)+r(B'B)\ge 4$. (Actually we have $r(A)=r(A'A)$ by SVD). Therefore, $r(A)+r(B)=4$ and $r(B)=4-s$. Let $A=U_0D_0V_0'$ and $B=U_1D_1V_1'$ both in SVD form, then we can take the singular matrices as $D_0=diag(d_0,\cdots,d_{s-1},0,\cdots,0)$, $D_1=diag(0,\cdots,0,d_s,\cdots,d_{n-1})$, $d_i>0$, $0\le i\le n-1$.

(ii) From $I=A'A+B'B=V_0D_0^2V_0'+V_1D_1^2V_1'$, we get $D_0^2V_0'V_1+V_0'V_1D_1^2=V_0'V_1$. Set $P=V_0'V_1=(p_{ij})$, we have $D_0^2P+PD_1^2=P$, or $P'D_0^2P=I-D_1^2$. Hence,

$$(P'D_0^2P)_{i,j}=\sum_{k=0}^{s-1}d_k^2p_{ki}p_{kj},\ \ 0\le i,j\le 3$$

$$(P'D_0^2P)_{i,i}=\sum_{k=0}^{s-1}d_k^2p_{ki}^2\ge 0,\ \ 0\le i\le 3$$

So $d_i\le 1$, $s\le i\le 3$. Similarly, we have $PD_1^2P'=I-D_0^2$ to conclude that $d_i\le 1$, $0\le i\le s-1$. Thus, $0<d_i\le 1$, $0\le i\le 3$.

(iii) From $O=A'B=V_0D_0U_0'U_1D_1V_1'$, we have $D_0U_0'U_1D_1=O$. Set $Q=U_0'U_1=(q_{ij})$. Using notations of matrix blocks,

$$D_0QD_1=\begin{bmatrix}D_{00}&0\\0&0\end{bmatrix}\begin{bmatrix}Q_0&Q_1\\Q_2&Q_3\end{bmatrix}\begin{bmatrix}0&0\\0&D_{10}\end{bmatrix}=\begin{bmatrix}0&D_{00}Q_1D_{10}\\0&0\end{bmatrix}$$

$$(D_{00}Q_1D_{10})_{i,j}=d_iq_{i,j}d_j,\ \ 0\le i\le s-1, s\le j\le 3$$

It concludes that $Q_1=0$, or $q_{ij}=0$, $0\le i\le s-1$, $s\le j\le 3$. From $AB'=O$, we have $D_0PD_1=O$. Similarly we get $p_{ij}=0$, $0\le i\le s-1$, $s\le j\le 3$.

(iv) Because $p_{ij}=0$, $0\le i\le s-1$, $s\le j\le 3$, it concludes that $d_j=1$, $s\le j\le 3$ since

$$1-d_j^2=(P'D_0^2P)_{j,j}=\sum_{k=0}^{s-1}d_k^2p_{k,j}^2=0,\ \ s\le j\le 3$$

Also from $PD_1^2P'=I-D_0^2$, we get $d_i=1$, $0\le i\le s-1$ with

$$1-d_i^2=(PD_1^2P')_{i,i}=\sum_{k=s}^{3}d_k^2p_{i,k}^2=0,\ \ 0\le i\le s-1$$

In summary, $d_i=1$, $0\le i\le 3$.

(v) Also we need to derive the relationship between V_0 and V_1, and U_0 and U_1. From $P=V_0'V_1$, we get

$$P=\begin{bmatrix}P_0&0\\P_2&P_3\end{bmatrix}=V_0'V_1=\begin{bmatrix}V_{00}'&V_{02}'\\V_{01}'&V_{03}'\end{bmatrix}\begin{bmatrix}V_{10}&V_{11}\\V_{12}&V_{13}\end{bmatrix}$$
$$=\begin{bmatrix}V_{00}'V_{10}+V_{02}'V_{12}&V_{00}'V_{11}+V_{02}'V_{13}\\V_{01}'V_{10}+V_{03}'V_{12}&V_{01}'V_{11}+V_{03}'V_{13}\end{bmatrix}$$

It concludes that the first s columns of V_0 are orthogonal to the last $(4-s)$ columns of V_1. Similarly, the first s columns of U_0 are orthogonal to the last $(4-s)$ columns of U_1. □

Formula (9) shows that A is only related to the first s columns of U_0 and V_0, B only related to the last $(4-s)$ columns of U_1 and V_1. Therefore, how to choose the last $(4-s)$ columns of U_0 and V_0 makes no difference. Particularly, we could choose $U_0=U_1$ and $V_0=V_1$, while $P=Q=I$.

4.2 case 2: L=2 for any $M\ge 2$

In this case the constraint equation system is still (5) and (6), but with dimension M. Also, the solutions above are valid. The scale factor M controls the dimension of matrices in constraint equation set and solutions. So we ignore the difference in M later.

4.3 case 3: L>2

In this case, the constraint equation set consists of L matrices denoted as $\{A_0,A_1,\cdots,A_{L-1}\}$. For example, when M=4 and L=4, the constraints equations are

$$[1,1,1,1](A_0+A_1+A_2+A_3)=[2,0,0,0] \tag{11}$$

$$\begin{cases}A_0'A_3=A_0A_3'=0\\A_0'A_2+A_1'A_3=A_0A_2'+A_1A_3'=0\\A_0'A_1+A_1'A_2+A_2'A_3=A_0A_1'+A_1A_2'+A_2A_3'=0\\A_0'A_0+A_1'A_1+A_2'A_2+A_3'A_3=A_0A_0'+A_1A_1'+A_2A_2'+A_3A_3'=I\end{cases} \tag{12}$$

while (12) appears to be formed by shifting and intersection of two sequences of $\{A_0,A_1,\cdots,A_3\}$.

Though *Theorem* 4 is not valid for this case, we still get partial solutions by the ways described in *Theorem* 3. That is, the same U and V are employed in the SVD form for all the $\{A_0,A_1,\cdots,A_3\}$. Let $A_i=UD_iV'$, $0\le i\le 3$. It is easy to see that (12) is satisfied if

$$\begin{cases} D_0 D_3 = 0 \\ D_0 D_2 + D_1 D_3 = 0 \\ D_0 D_1 + D_1 D_2 + D_2 D_3 = 0 \\ D_0^2 + D_1^2 + D_2^2 + D_3^2 = I \end{cases} \tag{13}$$

It concludes that the diagonal entries of D_i are 0 or 1, and each D_i has just one 1 on the diagonal to avoid $A_i = 0$. In a similar manner, (8) is required to meet (11). The limitation of this method is, some A_i has to be 0 when $L \geq M$, so more general solutions are desirable to construct longer filters when $L \geq M$.

5. Vanishing moments 2 for *M*=4 and *L*=2

5.1 Constraint formula for scaling filter

Scaling filter $\{h_{0,n}\}$, or $\{h_n\}$ for simplicity, having 2 vanishing moments requires:

$$\sum_{n\in Z} h_n (-i)^n = \sum_{n\in Z} h_n (-1)^n = \sum_{n\in Z} h_n i^n = 0 \tag{14}$$

$$\sum_{n\in Z} nh_n (-i)^n = \sum_{n\in Z} nh_n (-1)^n = \sum_{n\in Z} nh_n i^n = 0 \tag{15}$$

Formula (14) is always true since any scaling filter has at least 1 vanishing moment. Formula (15) is shift-invariant if the first condition becomes true:

$$\begin{aligned} &\sum_{n\in Z} (n+p) h_n (-i)^{n+p} = (-i)^p \Big(\sum_{n\in Z} nh_n (-i)^n + p \sum_{n\in Z} h_n (-i)^n\Big) \\ &= (-i)^p \sum_{n\in Z} nh_n (-i)^n \end{aligned}$$

For *M*=4 and *L*=2, formula (15) is equivalent to

$$\begin{bmatrix} 0 & i & -2 & -3i & 4 & 5i & -6 & -7i \\ 0 & -1 & 2 & -3 & 4 & -5 & 6 & -7 \\ 0 & -i & -2 & 3i & 4 & -5i & -6 & 7i \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \end{bmatrix}$$

$$= \begin{bmatrix} (-2h_2 + 4h_4 - 6h_6) + i(h_1 - 3h_3 + 5h_5 - 7h_7) \\ -h_1 + 2h_2 - 3h_3 + 4h_4 - 5h_5 + 6h_6 - 7h_7 \\ (-2h_2 + 4h_4 - 6h_6) - i(h_1 - 3h_3 + 5h_5 - 7h_7) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we have

$$\begin{cases} h_2 - 2h_4 + 3h_6 = 0 \\ h_1 - 3h_3 + 5h_5 - 7h_7 = 0 \\ -h_1 + 2h_2 - 3h_3 + 4h_4 - 5h_5 + 6h_6 - 7h_7 = 0 \end{cases} \tag{16}$$

If take h_4, h_5, h_6, h_7 as free parameters, we get

$$\begin{cases} h_0 = 2 - \frac{25}{3} h_4 + 4h_5 + 2h_6 + \frac{4}{3} h_7 \\ h_1 = 4h_4 - 5h_5 \\ h_2 = 2h_4 - 3h_6 \\ h_3 = \frac{4}{3} h_4 - \frac{7}{3} h_7 \end{cases}$$

5.2 Scaling filter construction

Let $A = UD_0V'$ and $B = UD_1V'$ as defined in Theorem 3, $D_0 = diag(d_{00}, d_{01}, d_{02}, d_{03})$, and $D_1 = diag(d_{10}, d_{11}, d_{12}, d_{13}) = I - D_0$, where $d_{0i} = 0 \text{ or } 1$, from (8) we have

$$v_{0k} = \frac{1}{2} \sum_{n=0}^{3} u_{nk}, \qquad 0 \leq k \leq 3 \tag{17}$$

The scaling filter, i.e. the first column of *A* and *B*, can be written with u_{ij} as:

$$\begin{cases} h_i = \sum_{k=0}^{3} d_{0k} u_{ik} v_{0k} = \frac{1}{2} \sum_{k=0}^{3} \sum_{n=0}^{3} d_{0k} u_{ik} u_{nk}, & 0 \leq i \leq 3 \\ h_{i+4} = \sum_{k=0}^{3} d_{1k} u_{ik} v_{0k} = \frac{1}{2} \sum_{k=0}^{3} \sum_{n=0}^{3} d_{1k} u_{ik} u_{nk}, & 0 \leq i \leq 3 \end{cases} \tag{18}$$

Case 1: $d_{00} = 0, d_{01} = d_{02} = d_{03} = 1$

If substitute (18) into (16), we get

$$\begin{cases} \sum_{k=0}^{3} v_{0k} \left[d_{0k} (2u_{2k}) + d_{1k} (-4u_{0k} + 6u_{2k}) \right] = 0 \\ \sum_{k=0}^{3} v_{0k} \left[d_{0k} (u_{1k} - 3u_{3k}) + d_{1k} (5u_{1k} - 7u_{3k}) \right] = 0 \\ \sum_{k=0}^{3} v_{0k} \left[d_{0k} (-u_{1k} + 2u_{2k} - 3u_{3k}) + d_{1k} (4u_{0k} - 5u_{1k} + 6u_{2k} - 7u_{3k}) \right] = 0 \end{cases} \tag{19}$$

Then we have

$$\begin{cases} \sum_{n=0}^{3} u_{n0} (-4u_{00} + 6u_{20}) + \sum_{k=1}^{3} \sum_{n=0}^{3} u_{nk} (2u_{2k}) = 0 \\ \sum_{n=0}^{3} u_{n0} (5u_{10} - 7u_{30}) + \sum_{k=1}^{3} \sum_{n=0}^{3} u_{nk} (u_{1k} - 3u_{3k}) = 0 \\ \sum_{n=0}^{3} u_{n0} (4u_{00} - 5u_{10} + 6u_{20} - 7u_{30}) + \sum_{k=1}^{3} \sum_{n=0}^{3} u_{nk} (-u_{1k} + 2u_{2k} - 3u_{3k}) = 0 \end{cases}$$

Because

$$\begin{aligned} &\sum_{k=1}^{3} \sum_{n=0}^{3} u_{nk} u_{ik} = \sum_{n=0}^{3} \Big(\sum_{k=0}^{3} u_{nk} u_{ik} - u_{n0} u_{i0}\Big) \\ &= \sum_{n=0}^{3} (\delta_{n,i} - u_{n0} u_{i0}) = 1 - \sum_{n=0}^{3} u_{n0} u_{i0}, \quad 0 \leq i \leq 3 \end{aligned}$$

we have

$$\begin{cases} (-u_{00} + u_{20}) \sum_{n=0}^{3} u_{n0} = -\frac{1}{2} \\ (u_{10} - u_{30}) \sum_{n=0}^{3} u_{n0} = \frac{1}{2} \\ (u_{00} - u_{10} + u_{20} - u_{30}) \sum_{n=0}^{3} u_{n0} = \frac{1}{2} \end{cases}$$

Then it concludes that

$$\begin{cases} \sum_{n=0}^{3} u_{n0}^2 = 1 \\ (-u_{00} + u_{20}) \sum_{n=0}^{3} u_{n0} = -\frac{1}{2} \\ 2u_{20} = u_{10} + u_{30} \\ u_{00} + u_{30} = u_{10} + u_{20} \end{cases}$$

Let $a = u_{00}, b = u_{10}, c = u_{20}, d = u_{30}$, then

$$\begin{cases} a^2 + b^2 + c^2 + d^2 = 1 \\ (-a + c)(a + b + c + d) = -\frac{1}{2} \\ 2c = b + d \\ a + d = b + c \end{cases}$$

$$\begin{cases} a^2+c^2+(\frac{a+c}{2})^2+(\frac{3c-a}{2})^2=1 \\ (a-c)(a+3c)=\frac{1}{2} \\ b=\frac{a+c}{2} \\ d=\frac{3c-a}{2} \end{cases}$$

We have a simplified equation system involving a and c:

$$\begin{cases} a^2+c^2=\frac{5}{8} \\ 2a^2+4ac-6c^2=1 \end{cases}$$

To solve it, we find four solutions, which are shown in Table 1. Then we can compute scaling filter $\{h_n\}_{0\le n\le 7}$ as:

$$\begin{cases} h_i=\frac{1}{2}(1-u_{i0}\sum_{n=0}^{3}u_{n0}), & 0\le i\le 3 \\ h_{i+4}=\frac{1}{2}-h_i, & 0\le i\le 3 \end{cases}$$

For four solutions of a, b, c and d, however, results of $\{h_n\}_{0\le n\le 7}$ are the same as the first column of Table 2, except for the difference in the reverse order.

Case 2: $d_{00}=d_{01}=0, d_{02}=d_{03}=1$

Let $\alpha=\sum_{n=0}^{3}u_{n2}$, $\beta=\sum_{n=0}^{3}u_{n3}$. From (19) we get

$$\begin{cases} (u_{02}-u_{22})\alpha+(u_{03}-u_{23})\beta=-\frac{1}{2} \\ (u_{12}-2u_{22}+u_{32})\alpha+(u_{13}-2u_{23}+u_{33})\beta=0 \\ (u_{02}-u_{12}-u_{22}+u_{32})\alpha+(u_{03}-u_{13}-u_{23}+u_{33})\beta=0 \end{cases}$$

Scaling filters can be computed from the last two columns of U:

$$\begin{cases} h_i=\frac{1}{2}(u_{i2}\alpha+u_{i3}\beta), & 0\le i\le 3 \\ h_{i+4}=\frac{1}{2}-h_i, & 0\le i\le 3 \end{cases}$$

We can compute the last two columns of U to obtain scaling filters. However, it seems that the scaling filters obtained are the same as the first column of Table 2.

Case 3: $d_{00}=d_{01}=d_{02}=0, d_{03}=1$

Similar to case 1, we get

$$\begin{cases} (u_{03}-u_{23})\sum_{n=0}^{3}u_{n3}=-\frac{1}{2} \\ (u_{13}-u_{33})\sum_{n=0}^{3}u_{n3}=-\frac{1}{2} \\ (u_{03}-u_{13}+u_{23}-u_{33})\sum_{n=0}^{3}u_{n3}=-\frac{1}{2} \end{cases}$$

Let $a=u_{03}, b=u_{13}, c=u_{23}, d=u_{33}$, then

$$\begin{cases} a^2+c^2=\frac{3}{8} \\ 2a^2+4ac-6c^2+1=0 \\ b=\frac{a+c}{2} \\ d=\frac{3c-a}{2} \end{cases}$$

Scaling filter $\{h_n\}_{0\le n\le 7}$ can be computed as follows:

$$\begin{cases} h_i=\frac{1}{2}u_{i3}\sum_{n=0}^{3}u_{n3}, & 0\le i\le 3 \\ h_{i+4}=\frac{1}{2}-h_i, & 0\le i\le 3 \end{cases}$$

In like manner, no new scaling filters are found for this case.

6. Conclusions

This paper gives an algebraic approach to construct M-band orthonormal wavelet bases, which enable us to obtain innumerable wavelet bases for selection. Also we give another way to compute 2-regular M-band filter banks. Our future work is to compute longer filter banks, to integrate the linear-phase property if possible, and to find better filters for wavelet-based image coding.

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Table 1: 4 solutions of *a, b, c, d* in case 1

a	-0.6742287	0.7876647	0.6742287	-0.7876647
b	-0.5435215	0.3599783	0.5435215	-0.3599783
c	-0.4128143	-0.0677081	0.4128143	0.0677081
d	-0.2821070	-0.4953944	0.2821070	0.4953944

Table 2: filters for M=4, L=2 with 2 vanishing moments

h_{0n}	h_{1n}	h_{2n}	h_{3n}
0.2697890	-0.2825435	0.4125840	0.2382055
0.3947890	0.5553379	-0.6279376	0.1088646
0.5197890	0.2385187	0.3727824	-0.7275830
0.6447890	-0.0783004	0.1487574	0.5572896
0.2302110	-0.5834819	-0.4125840	-0.2382055
0.1052110	-0.2666627	-0.1885590	-0.1088646
-0.0197890	0.0501564	0.0354659	0.0204763
-0.1447890	0.3669755	0.2594909	0.1498171